A new extension of the class of regularly varying functions

Meitner Cadena, Marie Kratz

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Meitner Cadena* and Marie Kratz†

Abstract

We define a new class of positive and Lebesgue measurable functions in terms of their asymptotic behavior, which includes the class of regularly varying functions. We also characterize it by transformations, corresponding to generalized moments when these functions are random variables. We study the properties and extensions of classical theorems for this class.

Keywords: asymptotic behavior; Karamata's representation theorem; Karamata's theorem; Karamata's tauberian theorem; measurable functions; Peter and Paul distribution; regularly varying function

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Introduction

The class of regularly varying functions has been introduced in the 30s by Karamata, who defined the notion of slowly varying (SV) and regularly varying (RV) functions, describing a specific asymptotic behavior of these functions, namely:
Definition. A Lebesgue-measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is RV at infinity if, for all $t > 0$,

$$\lim_{x\to\infty} \frac{U(xt)}{U(x)} = t^{\rho} \quad \text{for some } \rho \in \mathbb{R}, \tag{1}$$

ρ being called the tail index of U, and the case $\rho = 0$ corresponding to the notion of SV function. U is RV at 0^+ if (1) holds, when taking the limit as $x \to 0^+$ instead of $+\infty$.

Since then, much literature has been devoted to RV functions (see e.g. [33], [5] and references therein), in particular in Extreme Value Theory (EVT) (see e.g. [20], [18], [14], [31]) where the RV property helps characterizing maximum domains of attraction. The notion of multivariate regular variation has been developed (see e.g. [15], [32], and references therein) and various extensions of the RV class have been proposed. We may cite, in a non exhaustive way, the class of Extended RV (*ERV*) (which is implicit in the work of Matuszewska [29], and simply allows the limit in (1) to vary), its natural extension, named the

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O-Regularly Varying (*O-RV*) class, defined and studied by Avakumović [2], also analyzed by Karamata [27] (see also e.g. [29], [19], [33], [1], [28], and [11], where relations between *ERV* and *O-RV* are analyzed), the Bojanic-Karamata class (see [7]) that is a subclass of the SV class, the Π classes (see e.g. [3], or [5]), and the Beurling classes, the slowly varying one (see e.g. [5]) that contains the SV class, or the RV one (see [6]). It is worth noticing that the Beurling theory includes the Karamata theory (see [6]).

In this paper, we propose a new extension of the RV class, defined in terms of the asymptotic decay of the functions, and for which the limit in (1) might not exist. This new class not only extends in a simple way main RV properties but also offers broader applications, as e.g. in EVT. We can mention, for instance, new results on maximum domains domain of attraction (see [10]) and the proposition of a new tail index estimator (see [9]).

The aim of this work is to present and characterize fully this new class.

The paper is organized in two main parts. The first section defines this large class of functions, describing it in terms of their asymptotic behaviors, which may violate (1). It provides its algebraic properties, as well as characteristic representation theorems, one being of Karamata type. In the second section, we discuss extensions for this class of functions of other important Karamata theorems. Proofs of the results are given in the appendix.

1 Study of a new class of functions

We focus on the new class $\mathscr{M}$ of positive and measurable functions with support $\mathbb{R}^+$, characterizing their behavior at ∞ with respect to polynomial functions. A number of properties of this class are studied and characterizations are provided. Further, variants of this class, considering asymptotic behaviors of exponential type instead of polynomial one, provide other classes, denoted by $\mathscr{M}_\infty$ and $\mathscr{M}_{-\infty}$, having similar properties and characterizations as $\mathscr{M}$ does.

Let us introduce a few notations.

When considering limits, we will discriminate between two main cases, namely when the limit is finite or infinite ($\pm\infty$), and when it does not exist.

The notation a.s. (almost surely) in (in)equalities concerning measurable functions is omitted. Moreover, for any random variable (rv) X, we denote its distribution by $F_X(x) = P(X \le x)$, and its tail of distribution by $\overline{F}_X = 1 - F_X$. The subscript X will be omitted when no possible confusion.

RV (RV$_\rho$ respectively) denotes indifferently the class of regularly varying functions (with tail index ρ, respectively) or the property of regularly varying function (with tail index ρ).

Finally recall the notations $\min(a,b) = a \wedge b$ and $\max(a,b) = a \vee b$ that will be used, $\lfloor x \rfloor$ for the largest integer not greater than x and $\lceil x \rceil$ for the lowest integer greater or equal than x, and $\log(x)$ represents the natural logarithm of x.

1.1 The class $\mathscr{M}$

We introduce a new class $\mathscr{M}$ that we define as follows.

Definition 1.1. *$\mathcal{M}$ is the class of positive and measurable functions U with support $\mathbb{R}^+$, bounded on finite intervals, such that*

$$\exists\, \rho \in \mathbb{R}, \forall \varepsilon > 0, \lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty. \tag{2}$$

On $\mathcal{M}$, we can define specific properties.

Properties 1.1.

(i) *For any $U \in \mathcal{M}$, ρ defined in (2) is unique, and denoted by ρ_U.*

(ii) *If $U, V \in \mathcal{M}$ s.t. $\rho_U > \rho_V$, then $\lim_{x\to\infty} \frac{V(x)}{U(x)} = 0$.*

(iii) *For any $U, V \in \mathcal{M}$ and any $a \geq 0$, $aU + V \in \mathcal{M}$ with $\rho_{aU+V} = \rho_U \vee \rho_V$.*

(iv) *If $U \in \mathcal{M}$ with ρ_U defined in (2), then $1/U \in \mathcal{M}$ with $\rho_{1/U} = -\rho_U$.*

(v) *Let $U \in \mathcal{M}$ with ρ_U defined in* (2). *If $\rho_U < -1$, then U is integrable on $\mathbb{R}^+$, whereas, if $\rho_U > -1$, U is not integrable on $\mathbb{R}^+$.*
Note that in the case $\rho_U = -1$, we can find examples of functions U which are integrable or not.

(vi) Sufficient condition for U to belong to $\mathcal{M}$: *Let U be a positive and measurable function with support $\mathbb{R}^+$, bounded on finite intervals. Then*

$$-\infty < \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} < \infty \quad \Longrightarrow \quad U \in \mathcal{M}.$$

To simplify the notation, when no confusion is possible, we will denote ρ_U by ρ.

Remark 1.1. *Link to the notion of stochastic dominance*

Let X and Y be rv's with distributions F_X and F_Y, respectively, with support $\mathbb{R}^+$. We say that X is smaller than Y in the usual stochastic order (see e.g. [34], pp. 3) if

$$\overline{F}_X(x) \leq \overline{F}_Y(x) \quad \textit{for all } x \in \mathbb{R}^+. \tag{3}$$

This relation is also interpreted as the first-order stochastic dominance of X over Y, as $F_X \geq F_Y$ (see e.g. [22], pp. 289).

Let X, Y be rv's such that $\overline{F}_X = U$ and $\overline{F}_Y = V$, where $U, V \in \mathcal{M}$ and $\rho_U > \rho_V$. Then Properties 1.1, (ii), implies that there exists $x_0 > 0$ such that, for any $x \geq x_0$, $V(x) < U(x)$, hence that (3) *is satisfied at infinity, i.e. that X strictly dominates Y at infinity.*

Furthermore, the previous proof shows that a relation like (3) *is satisfied at infinity for any functions U and V in $\mathcal{M}$ satisfying $\rho_U > \rho_V$. It means that the notion of first-order stochastic dominance or stochastic order confined to rv's can be extended to functions in $\mathcal{M}$. In this way, we can say that if $\rho_U > \rho_V$, then U strictly dominates V at infinity.*

Now let us define, for any positive and measurable function U with support $\mathbb{R}^+$,

$$\kappa_U := \sup\left\{r : r \in \mathbb{R} \ \text{ and } \int_1^\infty x^{r-1} U(x)\,dx < \infty\right\}. \tag{4}$$

Note that κ_U may take values $\pm\infty$.

Definition 1.2. *For $U \in \mathcal{M}$, κ_U defined in* (4) *is called the $\mathcal{M}$-index of U.*

Remark 1.2.

1. *If the function U considered in* (4) *is bounded on finite intervals, then the integral involved can be computed on any interval $[a,\infty)$ with $a > 1$.*
2. *When assuming $U = \overline{F}$, F being a continuous distribution, the integral in (4) reduces (by changing the order of integration), for $r > 0$, to an expression of moment of a rv:*

$$\int_1^\infty x^{r-1}\overline{F}(x)\,dx = \frac{1}{r}\int_1^\infty \left(x^r - 1\right) dF(x) = \frac{1}{r}\int_1^\infty x^r\,dF(x) - \frac{\overline{F}(1)}{r}.$$

3. *We have $\kappa_U \geq 0$ for any tail $U = \overline{F}$ of a distribution F.*

   Indeed, suppose there exists $\overline{F}$ such that $\kappa_{\overline{F}} < 0$. Let us denote $\kappa_{\overline{F}}$ by κ. Since $\kappa < \kappa/2 < 0$, we have by definition of κ that $\displaystyle\int_1^\infty x^{\kappa/2-1}\overline{F}(x)\,dx = \infty$. But, since $\overline{F} \leq 1$ and $\kappa/2 - 1 < -1$, we can also write that $\displaystyle\int_1^\infty x^{\kappa/2-1}\overline{F}(x)\,dx \leq \int_1^\infty x^{\kappa/2-1}\,dx < \infty$. Hence the contradiction.

4. *A similar statement to Properties 1.1, (iii), has been proved for RV functions (see [5], pp. 16).*

Let us develop a simple example, also useful for the proofs.

***Example* 1.1.** *Let $\alpha \in \mathbb{R}$ and U_α the function defined on $(0,\infty)$ by*

$$U_\alpha(x) := \begin{cases} 1, & 0 < x < 1 \\ x^\alpha, & x \geq 1. \end{cases}$$

Then $U_\alpha \in \mathcal{M}$ with $\rho_{U_\alpha} = \alpha$ defined in (2), and its $\mathcal{M}$-index satisfies $\kappa_{U_\alpha} = -\alpha$.

To check that $U_\alpha \in \mathcal{M}$, it is enough to find a ρ_{U_α}, since its unicity follows by Properties 1.1, (i). Choosing $\rho_{U_\alpha} = \alpha$, we obtain, for any $\epsilon > 0$, that

$$\lim_{x\to\infty} \frac{U_\alpha(x)}{x^{\rho_{U_\alpha}+\epsilon}} = \lim_{x\to\infty} \frac{1}{x^\epsilon} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U_\alpha(x)}{x^{\rho_{U_\alpha}-\epsilon}} = \lim_{x\to\infty} x^\epsilon = \infty.$$

Hence U_α satisfies (2) with $\rho_{U_\alpha} = \alpha$.
Now, noticing that

$$\int_1^\infty x^{s-1} U_\alpha(x)\,dx = \int_1^\infty x^{s+\alpha-1}\,dx < \infty \quad \Longleftrightarrow \quad s + \alpha < 0$$

then κ_{U_α} defined in (4) satisfies $\kappa_{U_\alpha} = -\alpha$. $\square$

As a consequence of the definition of the $\mathcal{M}$-index κ on $\mathcal{M}$, we can prove that Properties 1.1, (vi), is not only a sufficient but also a necessary condition, obtaining then a first characterization of $\mathcal{M}$.

Theorem 1.1. ***First characterization of $\mathcal{M}$***
Let U be a positive measurable function with support $\mathbb{R}^+$ and bounded on finite intervals. Then

$$U \in \mathcal{M} \text{ with } \rho_U = -\tau \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\tau, \tag{5}$$

where ρ_U is defined in (2).

***Example* 1.2.** *The function U defined by $U(x) = x^{\sin(x)}$ does not belong to $\mathcal{M}$ since the limit expressed in (5) does not exist .*

Other properties on $\mathcal{M}$ can be deduced from Theorem 1.1, namely:

Properties 1.2. *For $U, V \in \mathcal{M}$ with ρ_U and ρ_V defined in* (2)*, respectively, we have:*

(i) *The product $UV \in \mathcal{M}$ with $\rho_{UV} = \rho_U + \rho_V$.*

(ii) *If $\rho_U \le \rho_V < -1$ or $\rho_U < -1 < 0 \le \rho_V$, then the convolution $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$. If $-1 < \rho_U \le \rho_V$, then $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_U + \rho_V$+1.*

(iii) *If $\lim_{x\to\infty} V(x) = \infty$, then $U \circ V \in \mathcal{M}$ with $\rho_{U\circ V} = \rho_U \rho_V$.*

Remark 1.3. *A similar statement to Properties 1.2, (ii), has been proved when restricting the functions U and V to RV probability density functions, showing first $\lim_{x\to\infty} \dfrac{U * V(x)}{U(x) + V(x)} = 1$ (see [4], Theorem 1.1). In contrast, we propose a direct proof, under the condition of integrability of the function of $\mathcal{M}$ having the lowest ρ.*

When U and V are tails of distributions belonging to RV, with the same tail index, Feller ([18], Proposition, pp. 278-279) proved that the tail of the convolution of $1-U$ and $1-V$ also belongs to this class and has the same tail index as U and V.

We can give a second way to characterize $\mathcal{M}$ using κ_U defined in (4).

Theorem 1.2. ***Second characterization of $\mathcal{M}$***
If U is a positive measurable function with support $\mathbb{R}^+$, bounded on finite intervals, then

$$U \in \mathcal{M} \text{ with associated } \rho_U \quad \Longleftrightarrow \quad \kappa_U = -\rho_U \tag{6}$$

where ρ_U satisfies (2) *and κ_U satisfies* (4).

Here is another characterization of $\mathcal{M}$, of Karamata type.

Theorem 1.3. ***Representation Theorem of Karamata type for $\mathcal{M}$***

(i) Let $U \in \mathcal{M}$ with finite ρ_U defined in (2). There exist $b > 1$ and functions α, β and ϵ satisfying, as $x \to \infty$,

$$\alpha(x)/\log(x) \to 0, \qquad \epsilon(x) \to 1, \qquad \beta(x) \to \rho_U, \tag{7}$$

such that, for $x \geq b$,

$$U(x) = \exp\left\{\alpha(x) + \epsilon(x) \int_b^x \frac{\beta(t)}{t} dt\right\}. \tag{8}$$

(ii) Conversely, if there exists a positive measurable function U with support $\mathbb{R}^+$, bounded on finite intervals, satisfying (8) for some $b > 1$ and functions α, β, and ϵ satisfying (7)*, then $U \in \mathcal{M}$ with finite ρ_U defined in (2).*

Remark 1.4.

1. *Another way to express* (8) *is the following:*

$$U(x) = \exp\left\{\alpha(x) + \frac{\epsilon(x)\log(x)}{x} \int_b^x \beta(t) dt\right\}. \tag{9}$$

2. *The function α defined in Theorem 1.3 is not necessarily bounded, contrarily to the case of Karamata representation for RV functions.*

***Example* 1.3.** *Let $U \in \mathcal{M}$ with $\mathcal{M}$-index κ_U. If there exists $c > 0$ such that $U < c$, then* $\kappa_U \geq 0$.

Indeed, since we have $\lim\limits_{x\to\infty} \dfrac{\log(1/U(x))}{\log(x)} \geq \lim\limits_{x\to\infty} \dfrac{\log(1/c)}{\log(x)} = 0$, applying Theorem 1.1 allows one to conclude. □

1.2 Extension of the class $\mathcal{M}$

We extend the class $\mathcal{M}$ introducing two other classes of functions.

Definition 1.3. *$\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ are the classes of positive measurable functions U with support $\mathbb{R}^+$, bounded on finite intervals, defined as*

$$\mathcal{M}_\infty := \left\{U \,:\, \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = 0\right\}, \tag{10}$$

and

$$\mathcal{M}_{-\infty} := \left\{U \,:\, \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = \infty\right\}. \tag{11}$$

Notice that it would be enough to consider $\rho < 0$ ($\rho > 0$, respectively) in (10) ((11), respectively), and that $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$ and $\mathcal{M}$ are disjoint.

We denote by $\mathcal{M}_{\pm\infty}$ the union $\mathcal{M}_\infty \cup \mathcal{M}_{-\infty}$.

We obtain similar properties for $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$, as the ones given for $\mathcal{M}$, namely:

Properties 1.3.

(i) $U \in \mathcal{M}_\infty \iff 1/U \in \mathcal{M}_{-\infty}$.

(ii) *If* $(U,V) \in \mathcal{M}_{-\infty} \times \mathcal{M}$ *or* $\mathcal{M}_{-\infty} \times \mathcal{M}_\infty$ *or* $\mathcal{M} \times \mathcal{M}_\infty$, *then* $\lim_{x\to\infty} \frac{V(x)}{U(x)} = 0$.

(iii) *If* $U, V \in \mathcal{M}_\infty$ *(*$\mathcal{M}_{-\infty}$ *respectively), then* $U+V \in \mathcal{M}_\infty$ *(*$\mathcal{M}_{-\infty}$ *respectively).*

The index κ_U defined in (4) may also be used to analyze $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$. It can take infinite values, as can be seen in the following example.

***Example* 1.4.** *Consider* U *defined on* $\mathbb{R}^+$ *by* $U(x) := e^{-x}$. *Then* $U \in \mathcal{M}_\infty$ *with* $\kappa_U = \infty$. *Choosing* $U(x) = e^x$ *leads to* $U \in \mathcal{M}_{-\infty}$ *with* $\kappa_U = -\infty$.

A first characterization of $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ can be provided, as done for $\mathcal{M}$ in Theorem 1.1.

Theorem 1.4. ***First characterization of*** $\mathcal{M}_\infty$ ***and*** $\mathcal{M}_{-\infty}$
Let U *be a positive measurable function with support* $\mathbb{R}^+$, *bounded on finite intervals. Then we have*

$$U \in \mathcal{M}_\infty \iff \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\infty \tag{12}$$

and

$$U \in \mathcal{M}_{-\infty} \iff \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = \infty. \tag{13}$$

Remark 1.5. *Link to a result from Daley and Goldie.*

If we restrict $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$ *to tails of distributions, then combining Theorems 1.1 and 1.4 and Theorem 2 in [13] provides another characterization, namely*

$$U \in \mathcal{M} \cup \mathcal{M}_{\pm\infty} \iff X_U \in \mathcal{M}^{DG},$$

where X_U *is a rv with tail* U *and* $\mathcal{M}^{DG}$ *is the set of non-negative rv's* X *having the property introduced by Daley and Goldie (see [13], Definition 1.(a)) that*

$$\kappa(X \wedge Y) = \kappa(X) + \kappa(Y)$$

for independent rv's X *and* Y. *We notice that* $\kappa(X)$ *defined in [13] (called there the moment index) and applied to rv's, coincides with the* $\mathcal{M}$*-index of* U, *when* U *is the tail of the distribution of* X.

An application of Theorem 1.4 provides properties as in Properties 1.2, namely:

Properties 1.4.

(i) *If* $(U,V) \in \mathcal{M}_\infty \times \mathcal{M}_\infty$ *or* $\mathcal{M}_{\pm\infty} \times \mathcal{M}$ *or* $\mathcal{M}_{-\infty} \times \mathcal{M}_{-\infty}$, *then* $U \cdot V \in \mathcal{M}_\infty$ *or* $\mathcal{M}_{\pm\infty}$ *or* $\mathcal{M}_{-\infty}$, *respectively.*

(ii) *If* $(U,V) \in \mathcal{M}_\infty \times \mathcal{M}$ *with* $\rho_V \geq 0$ *or* $\rho_V < -1$, *then* $U * V \in \mathcal{M}$ *with* $\rho_{U*V} = \rho_V$.
If $(U,V) \in \mathcal{M}_\infty \times \mathcal{M}_\infty$, *then* $U * V \in \mathcal{M}_\infty$.
If $(U,V) \in \mathcal{M}_{-\infty} \times \mathcal{M}$ *or* $\mathcal{M}_{-\infty} \times \mathcal{M}_{\pm\infty}$, *then* $U * V \in \mathcal{M}_{-\infty}$.

(iii) *If $U \in \mathcal{M}_{\pm\infty}$ and $V \in \mathcal{M}$ such that $\lim_{x\to\infty} V(x) = \infty$ or $V \in \mathcal{M}_{-\infty}$, then $U \circ V \in \mathcal{M}_{\pm\infty}$.*

Looking for extending Theorems 1.2-1.3 to $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ provides the next results.

Theorem 1.5.

Let U be a positive measurable function with support $\mathbb{R}^+$, bounded on finite intervals, with κ_U defined in (4).

(i) *(a)* $U \in \mathcal{M}_\infty \implies \kappa_U = \infty$.

(b) *U continuous, $\lim_{x\to\infty} U(x) = 0$, and $\kappa_U = \infty \implies U \in \mathcal{M}_\infty$.*

(ii) *(a)* $U \in \mathcal{M}_{-\infty} \implies \kappa_U = -\infty$.

(b) *U continuous and non-decreasing, and $\kappa_U = -\infty \implies U \in \mathcal{M}_{-\infty}$.*

Remark 1.6.

1. *In (i)-(b), the condition $\kappa_U = \infty$ might appear intuitively sufficient to prove that $U \in \mathcal{M}_\infty$. This is not true, as we can see with the following example showing for instance that the continuity assumption is needed. Indeed, we can check that the function U defined on $\mathbb{R}^+$ by*

$$U(x) := \begin{cases} 1/x & \textit{if} \quad x \in \bigcup\limits_{n\in\mathbb{N}\setminus\{0\}} (n; n+1/n^n) \\ e^{-x} & \textit{otherwise}, \end{cases}$$

satisfies $\kappa_U = \infty$ and $\lim_{x\to\infty} U(x) = 0$, but is not continuous and does not belong to $\mathcal{M}_\infty$.

2. *The proof of (i)-(b) is based on an integration by parts, isolating the term $t^r U(t)$. The continuity of U is needed, otherwise we would end up with an infinite number of jumps of the type $U(t^+) - U(t^-) (\neq 0)$ on $\mathbb{R}^+$.*

Theorem 1.6. ***Representation Theorem of Karamata Type for $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$***

(i) *If $U \in \mathcal{M}_\infty$, then there exist $b > 1$ and a positive measurable function α satisfying*

$$\alpha(x)/\log(x) \underset{x\to\infty}{\to} \infty, \tag{14}$$

such that, $\forall x \geq b$,

$$U(x) = \exp\{-\alpha(x)\}. \tag{15}$$

(ii) *If $U \in \mathcal{M}_{-\infty}$, then there exist $b > 1$ and a positive measurable function α satisfying (14) such that, $\forall x \geq b$,*

$$U(x) = \exp\{\alpha(x)\}. \tag{16}$$

(iii) *Conversely, if there exists a positive function U with support $\mathbb{R}^+$, bounded on finite intervals, satisfying (15) or (16), respectively, for some positive function α satisfying (14), then $U \in \mathcal{M}_\infty$ or $U \in \mathcal{M}_{-\infty}$, respectively.*

1.3 On the complement set of $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$

Considering measurable functions $U : \mathbb{R}^+ \to \mathbb{R}^+$, we have, applying Theorems 1.1 and 1.4, that U belongs to $\mathcal{M}$, $\mathcal{M}_\infty$ or $\mathcal{M}_{-\infty}$ if and only if $\displaystyle\lim_{x\to\infty} \frac{\log(U(x))}{\log(x)}$ exists, finite or infinite. Using the notions (see for instance [5], pp. 73) of *lower order* of U, defined by

$$\mu(U) := \underline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)}, \tag{17}$$

and *upper order* of U, defined by

$$\nu(U) := \overline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)}, \tag{18}$$

we can rewrite this characterization simply by $\mu(U) = \nu(U)$.

Hence, the complement set of $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$ in the set of functions $U : \mathbb{R}^+ \to \mathbb{R}^+$, denoted by $\mathcal{O}$, can be written as

$$\mathcal{O} := \{U : \mathbb{R}^+ \to \mathbb{R}^+ \; : \mu(U) < \nu(U)\}.$$

This set is nonempty: $\mathcal{O} \neq \emptyset$, as we are going to see through examples.

Examples of functions U satisfying $\mu(U) < \nu(U)$ are not well-known. A non explicit one was given by Daley (see [12], pp. 34) when considering rv's with discrete support (see [13], pp. 831). We will provide a couple of explicit parametric examples of functions in $\mathcal{O}$ which include tails of distributions with discrete support. These functions can be extended easily to continuous positive functions not necessarily monotone, for instance adapting polynomials given by Karamata (see [25], pp. 70-71). These examples are more detailed in Appendix A.3.

***Example* 1.5.**

Let $\alpha > 0$, $\beta \in \mathbb{R}$ such that $\beta \neq -1$, and $x_a > 1$. Let us consider the increasing series defined by $x_n = x_a^{(1+\alpha)^n}$, $n \geq 1$, well-defined because $x_a > 1$. Note that $x_n \to \infty$ as $n \to \infty$.

The function U defined by

$$U(x) := \begin{cases} 1, & 0 \leq x < x_1 \\ x_n^{\alpha(1+\beta)}, & x \in [x_n, x_{n+1}), \quad \forall n \geq 1, \end{cases} \tag{19}$$

belongs to $\mathcal{O}$, with

$$\begin{cases} \mu(U) = \dfrac{\alpha(1+\beta)}{1+\alpha} \quad \textit{and} \quad \nu(U) = \alpha(1+\beta), & \textit{if } 1+\beta > 0 \\[2ex] \mu(U) = \alpha(1+\beta) \quad \textit{and} \quad \nu(U) = \dfrac{\alpha(1+\beta)}{1+\alpha}, & \textit{if } 1+\beta < 0. \end{cases}$$

Moreover, if $1+\beta < 0$, then U is a tail of distribution whose associated rv has moments lower than $-\alpha(1+\beta)\big/(1+\alpha)$.

***Example* 1.6.**

Let $c > 0$ and $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$. Let $(x_n)_{n\in\mathbb{N}}$ be defined by $x_1 = 1$ and $x_{n+1} = 2^{x_n/c}$, $n \geq 1$, well-defined for $c > 0$. Note that $x_n \to \infty$ as $n \to \infty$.

The function U defined by

$$U(x) := \begin{cases} 1 & 0 \leq x < x_1 \\ 2^{\alpha x_n} & x_n \leq x < x_{n+1}, \quad \forall n \geq 1, \end{cases}$$

belongs to $\mathcal{O}$, with

$$\begin{cases} \mu(U) = \alpha c \quad \text{and} \quad \nu(U) = \infty, & \text{if } \alpha > 0 \\ \\ \mu(U) = -\infty \quad \text{and} \quad \nu(U) = \alpha c, & \text{if } \alpha < 0. \end{cases}$$

Moreover, if $\alpha < 0$, then U is a tail of distribution whose associated rv has moments lower than $-\alpha c$.

2 Extension of RV results

In this section, well-known results and fundamental in Extreme Value Theory, as Karamata's relations and Karamata's Tauberian Theorem, are discussed on $\mathcal{M}$. A key tool for the extension of these standard results to $\mathcal{M}$ is the characterizations of $\mathcal{M}$ given in Theorems 1.1 and 1.2.

First notice the relation between the class $\mathcal{M}$ introduced in the previous section and the class RV defined in (1).

Proposition 2.1. *RV_ρ ($\rho \in \mathbb{R}$) is a strict subset of $\mathcal{M}$.*

The proof of this claim comes from the Karamata relation (see [26]) given, for all RV function U with index $\rho \in \mathbb{R}$, by

$$\lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = \rho, \tag{20}$$

which implies, using Properties 1.1, (vi), that $U \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_U = -\rho$. Moreover, RV $\neq \mathcal{M}$, noticing that, for $t > 0$, $\lim_{x\to\infty} \dfrac{U(tx)}{U(x)}$ does not necessarily exist, whereas it does for a RV function U. For instance the function defined on $\mathbb{R}^+$ by $U(x) = 2 + \sin(x)$, is not RV, but $\lim_{x\to\infty} \dfrac{\log(U(x))}{\log(x)} = 0$, hence $U \in \mathcal{M}$.

2.1 Karamata's Theorem

We will focus on Karamata's well-known theorem developed for RV (see [23] and e.g. [14], Theorem 1.2.1) to analyze its extension to $\mathcal{M}$. Let us recall it, borrowing the version given in [14].

Theorem 2.1. ***Karamata's Theorem ([23]; e.g. [14], Theorem 1.2.1)***

Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals. Then

(K1)

$$U \in RV_\rho,\ \rho > -1 \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)\,dt} = \rho + 1 > 0.$$

(K2)

$$U \in RV_\rho,\ \rho < -1 \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)\,dt} = -\rho - 1 > 0.$$

(K3) (i) $U \in RV_{-1} \quad \Longrightarrow \quad \lim_{x\to\infty} \dfrac{xU(x)}{\int_0^x U(t)\,dt} = 0.$

(ii) $U \in RV_{-1}$ *and* $\displaystyle\int_0^\infty U(t)\,dt < \infty \quad \Longrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)\,dt} = 0.$

Remark 2.1. *The converse of (K3), (i), is false in general. A counterexample can be given by the Peter and Paul distribution which satisfies* $\lim_{x\to\infty} \dfrac{xU(x)}{\int_x^\infty U(t)\,dt} = 0$ *but is not* RV_{-1}*. We return to this in more detail in § 2.1.2.*

Theorem 2.1 is based on the existence of certain limits. We can extend some of the results to $\mathcal{M}$, even when theses limits do not exist, replacing them by more general expressions.

2.1.1 Karamata's Theorem on $\mathcal{M}$

Let us introduce the following conditions, in order to state the generalization of the Karamata Theorem to $\mathcal{M}$:

$(C1r)$ $\quad \dfrac{x^r U(x)}{\int_b^x t^{r-1}U(t)\,dt} \in \mathcal{M}$ with $\mathcal{M}$-index 0, *i.e.* $\displaystyle\lim_{x\to\infty}\left(\frac{\log\left(\int_b^x t^{r-1}U(t)\,dt\right)}{\log(x)} - \frac{\log(U(x))}{\log(x)}\right) = r.$

$(C2r)$ $\quad \dfrac{x^r U(x)}{\int_x^\infty t^{r-1}U(t)\,dt} \in \mathcal{M}$ with $\mathcal{M}$-index 0, *i.e.* $\displaystyle\lim_{x\to\infty}\left(\frac{\log\left(\int_x^\infty t^{r-1}U(t)\,dt\right)}{\log(x)} - \frac{\log(U(x))}{\log(x)}\right) = r.$

Theorem 2.2. ***Generalization of the Karamata Theorem to $\mathcal{M}$***

Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-summable on finite intervals, and $b > 0$. We have, for $r \in \mathbb{R}$,

(K1)*

$$U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho + r > 0 \quad \Longleftrightarrow \quad \begin{cases} \displaystyle\lim_{x\to\infty} \frac{\log\left(\int_b^x t^{r-1}U(t)\,dt\right)}{\log(x)} = \rho + r > 0 \\ U \text{ satisfies } (C1r). \end{cases}$$

(K2)*

$$U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho + r < 0 \quad \Longleftrightarrow \quad \begin{cases} \displaystyle\lim_{x\to\infty} \frac{\log\left(\int_x^\infty t^{r-1}U(t)\,dt\right)}{\log(x)} = \rho + r < 0 \\ U \text{ satisfies } (C2r). \end{cases}$$

(K3)*

$$U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho + r = 0 \iff \begin{cases} \lim\limits_{x\to\infty} \dfrac{\log\left(\int_b^x t^{r-1} U(t)dt\right)}{\log(x)} = \rho + r = 0 \\ \\ U \text{ satisfies } (C1r). \end{cases}$$

This theorem provides then a fourth characterization of $\mathcal{M}$.

Note that if $r = 1$, we can assume $b \geq 0$, as in the original Karamata's Theorem.

Remark 2.2.

1. *Note that (K3*) provides an equivalence contrarily to (K3).*
2. *Assuming that U satisfies the conditions $(C2r)$ and*

$$\int_1^\infty t^r U(t)dt < \infty, \tag{21}$$

we can propose a characterization of $U \in \mathcal{M}$ with $\mathcal{M}$-index $(r+1)$, namely

$$U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } (r+1) \iff \lim_{x\to\infty} \frac{\log\left(\int_x^\infty t^r U(t)dt\right)}{\log(x)} = 0.$$

This is the generalization of (K3) in Theorem 2.1, providing not only a necessary condition but also a sufficient one for U to belong to $\mathcal{M}$, under the conditions $(C2r)$ and (21).

2.1.2 Illustration using Peter and Paul distribution

The Peter and Paul distribution is a typical example of a function which is not RV. It is defined by (see e.g. [21], pp. 440, [17], pp. 50, [16], pp. 82, or [30], pp. 101)

$$F(x) := 1 - \sum_{k\geq 1:\, 2^k > x} 2^{-k}, \quad x > 0. \tag{22}$$

Let us illustrate the characterization theorems when applied to the Peter and Paul distribution; we do it for instance for Theorems 1.1 and 2.2, proving that this distribution belongs to $\mathcal{M}$.

Proposition 2.2.

The Peter and Paul distribution does not belong to RV, but to $\mathcal{M}$ with $\mathcal{M}$-index 1.

This proposition can be proved using Theorem 1.1 or Theorem 2.2. To illustrate the application of these two theorems, we develop the proof here and not in the appendix.

(i) *Application of Theorem 1.1*

For $x \in [2^n; 2^{n+1})$ $(n \geq 0)$, we have, using (22), $\overline{F}(x) = \sum_{k \geq n+1} 2^{-k} = 2^{-n}$, from which we deduce that $\frac{n}{n+1} \leq -\frac{\log\left(\overline{F}(x)\right)}{\log(x)} < 1$, hence $\lim_{x\to\infty} \frac{\log\left(\overline{F}(x)\right)}{\log(x)} = -1$, which by Theorem 1.1 is equivalent to

$$\overline{F} \in \mathscr{M} \quad \text{with} \quad \mathscr{M}-\text{index } 1.$$

(ii) *Application of Theorem 2.2*

Let us prove that

$$\lim_{x\to\infty} \frac{\log\left(\int_b^x \overline{F}(t)dt\right)}{\log(x)} = 0.$$

Suppose $2^n \leq x < 2^{n+1}$ and consider $a \in \mathbb{N}$ such that $a < n$. Choose w.l.o.g. $b = 2^a$.

Then the Peter and Paul distribution (22) satisfies

$$\int_b^x \overline{F}(t)dt = \sum_{k=a}^{n-1} \int_{2^k}^{2^{k+1}} \overline{F}(t)dt + \int_{2^n}^x \overline{F}(t)dt = \sum_{k=a}^{n-1} 2^{-k}(2^{k+1}-2^k)+(x-2^n)2^{-n} = n-a+x2^{-n}-1.$$

Hence

$$\frac{\log(n-a+x2^{-n}-1)}{(n+1)\log(2)} \leq \frac{\log\left(\int_b^x \overline{F}(t)dt\right)}{\log(x)} \leq \frac{\log(n-a+x2^{-n}-1)}{n\log(2)},$$

and, since $1 \leq 2^{-n}x < 2$, we obtain $\lim_{x\to\infty} \frac{\log\left(\int_b^x \overline{F}(t)dt\right)}{\log(x)} = 0$.

Moreover, we have

$$\lim_{x\to\infty} \frac{\log\left(\frac{x\overline{F}(x)}{\int_b^x \overline{F}(t)dt}\right)}{\log(x)} = 1 + \lim_{x\to\infty} \frac{\log\left(\overline{F}(x)\right)}{\log(x)} - \lim_{x\to\infty} \frac{\log\left(\int_b^x \overline{F}(t)dt\right)}{\log(x)} = 1.$$

Theorem 2.2 allows one then to conclude that $\overline{F} \in \mathscr{M}$ with $\mathscr{M}$-index 1. □

Note that the original Karamata Theorem (Theorem 2.1) does not allow one to prove that the Peter and Paul distribution is RV or not, since the converse of (i) in (K3) does not hold, contrarily to Theorem 2.2. Indeed, although we can prove that

$$\lim_{x\to\infty} \frac{x\overline{F}(x)}{\int_b^x \overline{F}(t)dt} = \lim_{x,n\to\infty} \frac{x2^{-n}}{n-a+x2^{-n}-1} = 0,$$

Theorem 2.1 does not imply that $\overline{F}$ is RV_{-1}.

2.2 Karamata's Tauberian Theorem

Let us recall Karamata's well-known Tauberian Theorem which deals on Laplace-Stieltjes (L-S) transforms and RV functions.

The L-S transform of a positive, right continuous function U with support $\mathbb{R}^+$ and with local bounded variation, is defined by

$$\widehat{U}(s) := \int_{(0;\infty)} e^{-xs}dU(x), \quad s > 0. \tag{23}$$

Theorem 2.3. ***Karamata's Tauberian Theorem (see [24])***

If U is a non-decreasing right continuous function with support $\mathbb{R}^+$ and satisfying $U(0^+) = 0$, with finite L-S transform $\widehat{U}$, then, for $\alpha > 0$,

$$U \in RV_\alpha \text{ at infinity} \iff \widehat{U} \in RV_\alpha \text{ at } 0^+.$$

Now we present the main result of this subsection which extends only partly the Karamata Tauberian Theorem to $\mathcal{M}$.

Theorem 2.4.

Let U be a continuous function with support $\mathbb{R}^+$ and local bounded variation, satisfying $U(0^+) = 0$. Let g be defined on $\mathbb{R}^+$ by $g(x) = 1/x$. Then, for any $\alpha > 0$,

(i) $U \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha) \implies \widehat{U} \circ g \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha)$.

(ii) $\left\{ \begin{array}{l} \widehat{U} \circ g \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\alpha) \\ \text{and } \exists \eta \in [0;\alpha) \; : \; x^{-\eta}U(x) \text{ concave} \end{array} \right. \implies U \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha)$.

3 Conclusion

We introduced a new class of positive functions with support $\mathbb{R}^+$, denoted by $\mathcal{M}$, strictly larger than the class of RV functions at infinity. We extended to $\mathcal{M}$ some well-known results given on RV class, which in particular will help to expand EVT beyond RV. This class satisfies a number of algebraic and characteristic properties, and its members U are characterized by a unique real number, called the $\mathcal{M}$-index κ_U. Extensions to $\mathcal{M}$ of the Karamata Theorems were discussed. Four characterizations of $\mathcal{M}$ were provided, one of them being the extension to $\mathcal{M}$ of Karamata's well-known theorem restricted to RV class. Furthermore, the cases $\kappa_U = \infty$ and $\kappa_U = -\infty$ were analyzed and their corresponding classes, denoted by $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ respectively, were identified and studied, as done for $\mathcal{M}$. The three sets $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$ and $\mathcal{M}$ are disjoint. Explicit examples of functions not belonging to $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$ were given.

Note that any result obtained here can be applied to functions with finite support, i.e. finite endpoint x^*, by using the change of variable $y = 1/(x^* - x)$ for $x < x^*$.

This new class seems promising in terms of applications. Several have already been developed, as the ones mentioned in the introduction (see [10], [9]). Note also a study comparing the various extensions of the RV class, including this new class (see [8]).

Further investigation will concern a multivariate version of $\mathcal{M}$.

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A Proofs of results given in Section 1

A.1 Proofs of results concerning $\mathcal{M}$

Proof of Theorem 1.1. The sufficient condition given in Theorem 1.1 comes from Properties 1.1, (vi). So it remains to prove its necessary condition, namely that

$$\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = -\rho_U, \tag{24}$$

for $U \in \mathcal{M}$ with finite ρ_U defined in (2).

Let $\epsilon > 0$ and define V by

$$V(x) = \begin{cases} 1, & 0 < x < 1 \\ x^{\rho_U+\epsilon}, & x \geq 1 \end{cases}$$

Applying Example 1.1 with $\alpha = \rho_U + \epsilon$ with $\epsilon > 0$ implies that $\rho_V = \rho_U + \epsilon$, hence $\rho_V > \rho_U$. Using Properties 1.1, (ii), provides then that

$$\lim_{x\to\infty} \frac{U(x)}{V(x)} = \lim_{x\to\infty} \frac{U(x)}{x^{\rho_U+\epsilon}} = 0,$$

so, for $n \in \mathbb{N}^*$, there exists $x_0 > 1$ such for all $x \geq x_0$,

$$\frac{U(x)}{x^{\rho_U+\epsilon}} \leq \frac{1}{n}, \quad \textit{i.e.} \quad n\,U(x) \leq x^{\rho_U+\epsilon}.$$

Applying the logarithm function to this last inequality and dividing it by $-\log(x)$, $x \geq x_0$, gives $-\dfrac{\log(n)}{\log(x)} - \dfrac{\log(U(x))}{\log(x)} \geq -\rho_U - \epsilon$, hence $-\dfrac{\log(U(x))}{\log(x)} \geq -\rho_U - \epsilon$, and then $\underline{\lim}_{x\to\infty} -\dfrac{\log(U(x))}{\log(x)} \geq -\rho_U - \epsilon$.

We consider now the function

$$W(x) = \begin{cases} 1, & 0 < x < 1 \\ x^{\rho_U-\epsilon}, & x \geq 1 \end{cases}$$

with $\epsilon > 0$ and proceed in the same way to obtain that, for any $\epsilon > 0$, $\overline{\lim}_{x\to\infty} -\dfrac{\log(U(x))}{\log(x)} \leq -\rho_U + \epsilon$. Hence, $\forall \epsilon > 0$, we have

$$-\rho_U - \epsilon \leq \underline{\lim}_{x\to\infty} -\frac{\log(U(x))}{\log(x)} \leq \overline{\lim}_{x\to\infty} -\frac{\log(U(x))}{\log(x)} \leq -\rho_U + \epsilon$$

from which the result follows taking ϵ arbitrary. □

Now we introduce a lemma, on which the proof of Theorem 1.2 will be based.

Lemma A.1. *Let $U \in \mathscr{M}$ with associated $\mathscr{M}$-index κ_U defined in (4). Then necessarily $\kappa_U = -\rho_U$, where ρ_U is defined in (2).*

Proof of Lemma A.1. Let $U \in \mathscr{M}$ with $\mathscr{M}$-index κ_U given in (4) and ρ_U defined in (2). By Theorem 1.1, we have $\lim\limits_{x\to\infty} \dfrac{\log(U(x))}{\log(x)} = \rho_U$.

Hence, for all $\epsilon > 0$ there exists $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq x^{\rho_U+\epsilon}$.

Multiplying this last inequality by x^{r-1}, $r \in \mathbb{R}$, and integrating it on $[x_0;\infty)$, we obtain

$$\int_{x_0}^{\infty} x^{r-1} U(x)\,dx \leq \int_{x_0}^{\infty} x^{\rho_U+\epsilon+r-1}\,dx$$

which is finite if $r < -\rho_U - \epsilon$. Taking $\epsilon \downarrow 0$ then the supremum on r leads to $\kappa_U = -\rho_U$. $\square$

Proof of Theorem 1.2.

The necessary condition is proved by Lemma A.1. The sufficient condition follows from the assumption that ρ_U satisfies (2). $\square$

Proof of Theorem 1.3.

- *Proof of (i)*

  For $U \in \mathscr{M}$, Theorems 1.1 and 1.2 give that

$$\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = -\rho_U = \kappa_U \quad \text{with } \rho_U \text{ defined in (2) and } \kappa_U \text{ in (4).} \tag{25}$$

  Introducing a function γ such that

$$\lim_{x\to\infty} \gamma(x) = 0, \tag{26}$$

  we can write, for some $b > 1$, applying the L'Hôpital's rule to the ratio,

$$\lim_{x\to\infty} \left(\gamma(x) + \frac{\int_b^x \frac{\log(U(t))}{\log(t)} \frac{dt}{t}}{\log(x)} \right) = \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\kappa_U. \tag{27}$$

  ▹ Suppose $\kappa_U \neq 0$. Then we deduce from (25) and (27), that

$$\lim_{x\to\infty} \frac{\log(U(x))}{\gamma(x)\log(x) + \int_b^x \frac{\log(U(t))}{t\log(t)}\,dt} = 1. \tag{28}$$

  Hence, defining the function $\epsilon_U(x) := \dfrac{\log(U(x))}{\gamma(x)\log(x) + \int_b^x \frac{\log(U(t))}{t\log(t)}\,dt}$, for $x \geq b$, we can express U, for $x \geq b$, as

$$U(x) = \exp\left\{ \alpha_U(x) + \epsilon_U(x) \int_b^x \frac{\beta_U(t)}{t}\,dt \right\}$$

$$\text{where} \quad \alpha_U(x) := \epsilon_U(x)\,\gamma(x)\,\log(x) \quad \text{and} \quad \beta_U(x) := \frac{\log(U(x))}{\log(x)}. \tag{29}$$

It is then straightforward to check that the functions α_U, β_U and ϵ_U satisfy the conditions given in Theorem 1.3. Indeed, by (26) and (28), $\lim_{x\to\infty} \frac{\alpha_U(x)}{\log(x)} = \lim_{x\to\infty} \epsilon_U(x)\,\gamma(x) = 0$. Using (25), we obtain $\lim_{x\to\infty} \beta_U(x) = \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\kappa_U = \rho_U$. Finally, by (28), we have $\lim_{x\to\infty} \epsilon_U(x) = 1$.

▷ Now suppose $\kappa_U = 0$.

We want to prove (8) for some functions α, β, and ϵ satisfying (7).

Notice that (25) with $\kappa_U = 0$ allows one to write that $\lim_{x\to\infty} \frac{\log(x\,U(x))}{\log(x)} = 1$.

So applying Theorem 1.1 to the function V defined by $V(x) = xU(x)$, gives that $V \in \mathcal{M}$ with $\rho_V = -\kappa_V = 1$. Since $\kappa_V \neq 0$, we can proceed in the same way as previously, and obtain a representation for V of the form (8), namely, for $d > 1$, $\forall x \geq d$,

$$V(x) = \exp\left\{\alpha_V(x) + \epsilon_V(x) \int_d^x \frac{\beta_V(t)}{t}\,dt\right\}$$

where α_V, β_V, ϵ_V satisfy the conditions of Theorem 1.3 and $\beta_V = \frac{\log(V(x))}{\log(x)}$ (see (29)). Hence we have, for $x \geq d$,

$$\begin{aligned} U(x) &= \frac{V(x)}{x} = \exp\left\{-\log(x) + \alpha_V(x) + \epsilon_V(x) \int_d^x \frac{\log(t\,U(t))}{t\,\log(t)}\,dt\right\} \\ &= \exp\left\{\alpha_V(x) + (\epsilon_V(x) - 1)\log(x) - \epsilon_V(x)\log(d) + \epsilon_V(x) \int_d^x \frac{\log(U(t))}{t\,\log(t)}\,dt\right\}. \end{aligned}$$

Noticing that $\lim_{x\to\infty} \frac{\alpha_V(x) + (\epsilon_V(x)-1)\log(x) - \epsilon_V(x)\log(d)}{\log(x)} = 0$, we obtain that U satisfies (8) when setting, for $x \geq d$, $\alpha_U(x) := \alpha_V(x) + (\epsilon_V(x) - 1)\log(x) - \epsilon_V(x)\log(d)$, $\beta_U(x) := \frac{\log(U(x))}{\log(x)}$ and $\epsilon_U := \epsilon_V$.

- *Proof of (ii)*

  Let U be a positive function with support $\mathbb{R}^+$, bounded on finite intervals. Assume that U can be expressed as (8) for some functions α, β, and ϵ satisfying (7). We are going to check the sufficient condition given in Properties 1.1, (vi), to prove that $U \in \mathcal{M}$.

  Since $\frac{\log(U(x))}{\log(x)} = \frac{\alpha(x)}{\log(x)} + \epsilon(x)\frac{\int_b^x \frac{\beta(t)}{t}dt}{\log(x)}$ and that, via L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\int_b^x \frac{\beta(t)}{t}dt}{\log(x)} = \lim_{x\to\infty} \frac{\beta(x)/x}{1/x} = \lim_{x\to\infty} \beta(x),$$

  then using the limits of α, β, and ϵ allows one to conclude.

□

Proof of Properties 1.1.

- *Proof of (i)*

  Let us prove this property by contradiction.

  Suppose there exist ρ and ρ', with $\rho' < \rho$, both satisfying (2), for $U \in \mathcal{M}$. Choosing $\epsilon = (\rho - \rho')/2$ in (2) gives

  $$\lim_{x\to\infty} \frac{U(x)}{x^{\rho'+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \lim_{x\to\infty} \frac{U(x)}{x^{\rho'+\epsilon}} = \infty,$$

  hence the contradiction.

- *Proof of (ii)*

  Choosing $\epsilon = (\rho_U - \rho_V)/2$, we can write

  $$\frac{V(x)}{U(x)} = \frac{V(x)}{x^{\rho_V+\epsilon}} \frac{x^{\rho_V+\epsilon}}{U(x)} = \frac{V(x)}{x^{\rho_V+\epsilon}} \left(\frac{U(x)}{x^{\rho_U-\epsilon}}\right)^{-1},$$

  from which we deduce (ii).

- *Proof of (iii)*

  Let $U, V \in \mathcal{M}$, $a > 0$, $\epsilon > 0$ and suppose w.l.o.g. that $\rho_U \le \rho_V$.

  Since $\rho_V - \rho_U > 0$, writing $\dfrac{aU(x)}{x^{\rho_V \pm\epsilon}} = \dfrac{a}{x^{\rho_V-\rho_U}} \dfrac{U(x)}{x^{\rho_U\pm\epsilon}}$ gives $\lim\limits_{x\to\infty} \dfrac{aU(x)+V(x)}{x^{\rho_V+\epsilon}} = 0$ and $\lim\limits_{x\to\infty} \dfrac{aU(x)+V(x)}{x^{\rho_V-\epsilon}} = \infty$, we conclude thus that $\rho_{aU+V} = \rho_U \vee \rho_V$.

- *Proof of (iv)*

  It is straightforward since (2) can be rewritten as

  $$\lim_{x\to\infty} \frac{1/U(x)}{x^{-\rho_U-\epsilon}} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{1/U(x)}{x^{-\rho_U+\epsilon}} = 0.$$

- *Proof of (v)*

  First, let us consider $U \in \mathcal{M}$ with $\rho_U < -1$.

  Choosing $\epsilon_0 = -(\rho_U + 1)/2$ (> 0) in (2) implies that there exist $C > 0$ and $x_0 > 1$ such that, for $x \ge x_0$, $U(x) \le C\, x^{\rho_U+\epsilon_0} = C\, x^{(\rho_U-1)/2}$, from which we deduce that

  $$\int_{x_0}^{\infty} U(x)\, dx < \infty.$$

  We conclude that $\displaystyle\int_0^{\infty} U(x)\, dx < \infty$ because U is bounded on finite intervals.

  Now suppose that $\rho_U > -1$.

  Choosing $\epsilon_0 = (\rho_U + 1)/2$ (> 0) in (2) gives that for $C > 0$ there exists $x_0 > 1$ such that, for $x \ge x_0$, $U(x) \ge C\, x^{(\rho_U-1)/2}$ $\displaystyle\int_0^{\infty} U(x)\, dx \ge \int_{x_0}^{\infty} U(x)\, dx \ge \infty.$

- *Proof of (vi)*

  Assuming $-\infty < \lim\limits_{x\to\infty} \frac{\log(U(x))}{\log(x)} < \infty$, we want to prove that U satisfies (2), which implies that $U \in \mathcal{M}$.

  So let us prove (2).

  Consider $\rho = \lim\limits_{x\to\infty} \frac{\log(U(x))}{\log(x)}$ well defined under our assumption, and from which we can deduce that,

  $$\forall \epsilon > 0, \exists x_0 > 1 \text{ such that, } \forall x \geq x_0, \quad -\frac{\epsilon}{2} \leq \frac{\log(U(x))}{\log(x)} - \rho \leq \frac{\epsilon}{2}.$$

  Therefore we can write that, for $x \geq x_0$, on one hand,

  $$0 \leq \frac{U(x)}{x^{\rho+\epsilon}} = \exp\left\{\left(\frac{\log(U(x))}{\log(x)} - \rho - \epsilon\right)\log(x)\right\} \leq \exp\left\{-\frac{\epsilon}{2}\log(x)\right\} \underset{x\to\infty}{\longrightarrow} 0,$$

  and on the other hand,

  $$\frac{U(x)}{x^{\rho-\epsilon}} = \exp\left\{\left(\frac{\log(U(x))}{\log(x)} - \rho + \epsilon\right)\log(x)\right\} \geq \exp\left\{\frac{\epsilon}{2}\log(x)\right\} \underset{x\to\infty}{\longrightarrow} \infty,$$

  hence the result. □

Proof of Properties 1.2.

Let U, $V \in \mathcal{M}$ with ρ_U and ρ_V respectively, defined in (2).

- *Proof of (i)*

  It is immediate since

  $$\lim_{x\to\infty} \frac{\log(U(x)\,V(x))}{\log(x)} = \lim_{x\to\infty}\left(\frac{\log(U(x))}{\log(x)} + \frac{\log(V(x))}{\log(x)}\right) = \rho_U + \rho_V$$

- *Proof of (ii)*

  First notice that, since $U, V \in \mathcal{M}$, via Theorems 1.1 and 1.2, for $\epsilon > 0$, there exist $x_U > 0$, $x_V > 0$, such that, for $x \geq x_0 = x_U \vee x_V$,

  $$x^{\rho_U-\epsilon/2} \leq U(x) \leq x^{\rho_U+\epsilon/2} \quad \text{and} \quad x^{\rho_V-\epsilon/2} \leq V(x) \leq x^{\rho_V+\epsilon/2}.$$

  ▷ *Assume* $\rho_U \leq \rho_V < -1$. Hence, via Properties 1.1, (v), both U and V are integrable on $\mathbb{R}^+$. Choose $\rho = \rho_V$.

  Via the change of variable $s = x - t$, we have, $\forall\ x \geq 2x_0 > 0$,

  $$\begin{aligned}\frac{U*V(x)}{x^{\rho+\epsilon}} &= \int_0^{x/2} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt + \int_{x/2}^{x} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt \\ &\leq \frac{1}{x^{\epsilon/2}}\int_0^{x/2} U(t)\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x/2} V(s)\left(1-\frac{s}{x}\right)^{\rho_U+\epsilon/2} ds \\ &\leq \frac{\max\left(1, c^{\rho_V+\epsilon/2}\right)}{x^{\epsilon/2}}\int_0^{x/2} U(t)dt + \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x/2} V(s)ds,\end{aligned}$$

since, for $0 \le t \le x/2$, *i.e.* $0 < c < \dfrac{1}{2} \le 1 - \dfrac{t}{x} \le 1$,

$$\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} \le \max\left(1, c^{\rho_V+\epsilon/2}\right) \quad \text{and} \quad \left(1-\frac{t}{x}\right)^{\rho_U+\epsilon/2} \le \max\left(1, c^{\rho_U+\epsilon/2}\right).$$

Hence we obtain, U and V being integrable, and since $\rho_V - \rho_U + \epsilon/2 > 0$,

$$\lim_{x\to\infty} \frac{\max\left(1, c^{\rho_V+\epsilon/2}\right)}{x^{\epsilon/2}} \int_0^{x/2} U(t)dt = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}} \int_0^{x/2} V(s)ds = 0,$$

from which we deduce that, for any $\epsilon > 0$, $\displaystyle\lim_{x\to\infty} \frac{U * V(x)}{x^{\rho+\epsilon}} = 0$.

Applying Fatou's Lemma, then using that $V \in \mathcal{M}$ with $\rho_V = \rho$, gives, for any ϵ,

$$\lim_{x\to\infty} \frac{U*V(x)}{x^{\rho-\epsilon}} \ge \lim_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \ge \varliminf_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \ge \int_0^1 U(t) \lim_{x\to\infty} \left(\frac{V(x-t)}{x^{\rho-\epsilon}}\right) dt = \infty.$$

We can conclude that $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$.

▷ *Assume* $\rho_U < -1 < 0 \le \rho_V$. Therefore U is integrable on $\mathbb{R}^+$, but not V (Properties 1.1, (v)). Choose $\rho = \rho_V$.

Using the change of variable $s = x - t$, we have, $\forall\ x \ge 2x_0 > x_0 (> 0)$,

$$\begin{aligned}
\frac{U*V(x)}{x^{\rho+\epsilon}} &= \int_0^{x-x_0} U(t) \frac{V(x-t)}{x^{\rho+\epsilon}} dt + \int_{x-x_0}^{x} U(t) \frac{V(x-t)}{x^{\rho+\epsilon}} dt \\
&= \int_0^{x-x_0} U(t) \frac{V(x-t)}{x^{\rho+\epsilon}} dt + \int_0^{x_0} V(s) \frac{U(x-s)}{x^{\rho+\epsilon}} ds \\
&\le \int_0^{x-x_0} U(t) \frac{(x-t)^{\rho_V+\epsilon/2}}{x^{\rho+\epsilon}} dt + \int_0^{x_0} V(s) \frac{(x-s)^{\rho_U+\epsilon/2}}{x^{\rho+\epsilon}} ds \\
&= \frac{1}{x^{\epsilon/2}} \int_0^{x-x_0} U(t) \left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V-\rho_U+\epsilon/2}} \int_0^{x_0} V(s) \left(1-\frac{s}{x}\right)^{\rho_U+\epsilon/2} ds.
\end{aligned}$$

Noticing that for $0 \le t \le x - x_0$, so $\left(1-\dfrac{t}{x}\right)^{\rho_V+\epsilon/2} \le 1$, and for $0 \le s \le x_0 < 2x_0 \le x$, $0 < c < \dfrac{1}{2} \le 1 - \dfrac{x_0}{x} \le 1 - \dfrac{s}{x} \le 1$, so $\left(1-\dfrac{s}{x}\right)^{\rho_U+\epsilon/2} \le \max\left(1, c^{\rho_U+\epsilon/2}\right)$, we obtain

$$\frac{U*V(x)}{x^{\rho+\epsilon}} \le \frac{1}{x^{\epsilon/2}} \int_0^{x-x_0} U(t)dt + \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}} \int_0^{x_0} V(s)ds.$$

Since U is integrable, V bounded on finite intervals, and $\rho_V - \rho_U + \epsilon/2 > 0$, we have

$$\lim_{x\to\infty} \frac{1}{x^{\epsilon/2}} \int_0^{x-x_0} U(t)dt = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}} \int_0^{x_0} V(t)dt = 0.$$

therefore, for any $\epsilon > 0$, we have $\displaystyle\lim_{x\to\infty} \frac{U*V(x)}{x^{\rho+\epsilon}} = 0$.

Applying Fatou's Lemma, then using that $V \in \mathcal{M}$ with $\rho_V = \rho$, gives, for any ϵ,

$$\lim_{x\to\infty} \frac{U*V(x)}{x^{\rho-\epsilon}} \ge \lim_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \ge \varliminf_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \ge \int_0^1 U(t) \lim_{x\to\infty} \left(\frac{V(x-t)}{x^{\rho-\epsilon}}\right) dt = \infty.$$

We can conclude that $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$.

⊳ *Assume* $-1 < \rho_U \le \rho_V$. Then both U and V are not integrable on $\mathbb{R}^+$ (Properties 1.1, (v)). Choose $\rho = \rho_U + \rho_V + 1$.

Let $0 < \epsilon < \rho_U + 1$. Since V is not integrable on $\mathbb{R}^+$, we have $\displaystyle\int_0^x V(t)dt \underset{x\to\infty}{\to} \infty$. So we can apply the L'Hôpital's rule and obtain

$$\lim_{x\to\infty} \frac{\int_0^x V(t)dt}{x^{\rho_V+1+\epsilon}} = \lim_{x\to\infty} \frac{\left(\int_0^x V(t)dt\right)'}{\left(x^{\rho_V+1+\epsilon}\right)'} = \lim_{x\to\infty} \frac{V(x)}{(\rho_V+1+\epsilon)x^{\rho_V+\epsilon}} = 0$$

and

$$\lim_{x\to\infty} \frac{\int_0^x V(t)dt}{x^{\rho_V+1-\epsilon}} = \lim_{x\to\infty} \frac{\left(\int_0^x V(t)dt\right)'}{\left(x^{\rho_V+1-\epsilon}\right)'} = \lim_{x\to\infty} \frac{V(x)}{(\rho_V+1-\epsilon)x^{\rho_V-\epsilon}} = \infty,$$

from which we deduce that $W_V(x) := \displaystyle\int_0^x V(t)dt \in \mathcal{M}$ with $\mathcal{M}$-index $\rho_V + 1$.

We obtain in the same way that $W_U(x) := \displaystyle\int_0^x U(t)dt \in \mathcal{M}$ with $\mathcal{M}$-index $\rho_U + 1$.

We have, via the change of variable $s = x - t$, $\forall\ x \ge 2x_0 > 0$,

$$\begin{aligned}
\frac{U*V(x)}{x^{\rho+\epsilon}} &= \int_0^{x/2} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt + \int_{x/2}^x U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt \\
&\le \frac{1}{x^{\rho_U+1+\epsilon/2}}\int_0^{x/2} U(t)\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V+1+\epsilon/2}}\int_0^{x/2} V(s)\left(1-\frac{s}{x}\right)^{\rho_U+\epsilon/2} ds \\
&\le \max\left(1, c^{\rho_V+\epsilon/2}\right)\frac{W_U(x/2)}{x^{\rho_U+1+\epsilon/2}} + \max\left(1, c^{\rho_U+\epsilon/2}\right)\frac{W_V(x/2)}{x^{\rho_V+1+\epsilon/2}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{U*V(x)}{x^{\rho-\epsilon}} &= \int_0^{x/2} U(t)\frac{V(x-t)}{x^{\rho-\epsilon}}dt + \int_{x/2}^x U(t)\frac{V(x-t)}{x^{\rho-\epsilon}}dt \\
&\ge \frac{1}{x^{\rho_U+1-\epsilon/2}}\int_0^{x/2} U(t)\left(1-\frac{t}{x}\right)^{\rho_V-\epsilon/2} dt + \frac{1}{x^{\rho_V+1-\epsilon/2}}\int_0^{x/2} V(s)\left(1-\frac{s}{x}\right)^{\rho_U-\epsilon/2} ds \\
&\ge \min\left(1, c^{\rho_V-\epsilon/2}\right)\frac{W_U(x/2)}{x^{\rho_U+1-\epsilon/2}} + \min\left(1, c^{\rho_U-\epsilon/2}\right)\frac{W_V(x/2)}{x^{\rho_V+1-\epsilon/2}},
\end{aligned}$$

since, for $0 \le t \le x/2$, *i.e.* $0 < c < \dfrac{1}{2} \le 1 - \dfrac{t}{x} \le 1$,

$$\min\left(1, c^{\rho_V-\epsilon/2}\right) \le \left(1-\frac{t}{x}\right)^{\rho_V-\epsilon/2} \le \left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} \le \max\left(1, c^{\rho_V+\epsilon/2}\right)$$

and

$$\min\left(1, c^{\rho_U-\epsilon/2}\right) \le \left(1-\frac{t}{x}\right)^{\rho_U-\epsilon/2} \le \left(1-\frac{t}{x}\right)^{\rho_U+\epsilon/2} \le \max\left(1, c^{\rho_U+\epsilon/2}\right).$$

Hence, for any $0 < \epsilon < \rho_U + 1$, we have $\displaystyle\lim_{x\to\infty}\frac{U*V(x)}{x^{\rho+\epsilon}} = 0$ and $\displaystyle\lim_{x\to\infty}\frac{U*V(x)}{x^{\rho-\epsilon}} = \infty$. We can conclude that $U*V \in \mathcal{M}$ with $\rho_{U*V} = \rho_U + \rho_V + 1$.

- *Proof of (iii)*

  It is straightforward, since we can write, with $y = V(x) \to \infty$ as $x \to \infty$,

  $$\lim_{x\to\infty} \frac{\log(U(V(x)))}{\log(x)} = \lim_{y\to\infty} \frac{\log(U(y))}{\log(y)} \times \lim_{x\to\infty} \frac{\log(V(x))}{\log(x)} = \rho_U \, \rho_V$$

  Hence we obtain $\rho_{U\circ V} = \rho_U \, \rho_V$. $\square$

A.2 Proofs of results concerning $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$

Proof of Theorem 1.4.

It is enough to prove (12) because by this equivalence and Properties 1.3, (i), one has

$$U \in \mathcal{M}_{-\infty} \iff 1/U \in \mathcal{M}_\infty \iff \lim_{x\to\infty} -\frac{\log(1/U(x))}{\log(x)} = \infty \iff \lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = -\infty,$$

i.e. (13).

- Let us prove that $U \in \mathcal{M}_\infty \implies \lim\limits_{x\to\infty} \dfrac{\log(U(x))}{\log(x)} = -\infty$.

  Suppose $U \in \mathcal{M}_\infty$. This implies that for all $\rho \in \mathbb{R}$, one has $\lim\limits_{x\to\infty} \dfrac{U(x)}{x^\rho} = 0$, i.e. for all $\epsilon > 0$ there exists $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq \epsilon x^\rho$ which implies $\dfrac{\log(U(x))}{\log(x)} \leq \dfrac{\log(\epsilon)}{\log(x)} + \rho$, hence $\lim\limits_{x\to\infty} \dfrac{\log(U(x))}{\log(x)} \leq \rho$ and the statement follows since the argument applies for all $\rho \in \mathbb{R}$.

- Now let us prove that $\lim\limits_{x\to\infty} -\dfrac{\log(U(x))}{\log(x)} = \infty \implies U \in \mathcal{M}_\infty$.

  For any $\rho \in \mathbb{R}$, we can write

  $$\lim_{x\to\infty} -\frac{\log\left(\frac{U(x)}{x^\rho}\right)}{\log(x)} = \lim_{x\to\infty} \left(-\frac{\log(U(x))}{\log(x)} + \rho\right) = \infty \quad \text{under the hypothesis,}$$

  which implies that $U(x)/x^\rho < 1$ and hence $\lim\limits_{x\to\infty} \dfrac{U(x)}{x^\rho} = 0$. $\square$

Proof of Theorem 1.5.

- *Proof of (i)-(a)*

  Suppose $U \in \mathcal{M}_\infty$. Then, by definition (10), for any $\rho \in \mathbb{R}$, $\lim\limits_{x\to\infty} x^\rho U(x) = 0$, which implies that for $c > 0$, there exists $x_0 > 1$ such that, for all $x \geq x_0$, $U(x) \leq c x^{-\rho}$, from which we deduce that

  $$\int_{x_0}^{\infty} x^{r-1} U(x) dx \leq c \int_{x_0}^{\infty} x^{r-1-\rho} dx$$

  which is finite whenever $r < \rho$. This result holds also on $(1;\infty)$ since U is bounded on finite intervals.

  Thus we conclude that $\kappa_U = \infty$, ρ being any real number.

- *Proof of (i)-(b)*

  Note that U is integrable on $\mathbb{R}^+$ since $\int_1^\infty x^{r-1}U(x)dx<\infty$, for any $r\in\mathbb{R}$, in particular for $r=1$. Moreover U is bounded on finite intervals.

  For $r>0$, we have, via the continuity of U,

$$\int_0^\infty x^{r+1}dU(x)=(r+1)\int_0^\infty\int_0^x y^r\,dy\,dU(x)=(r+1)\int_0^\infty y^r\left(\int_y^\infty dU(x)\right)dy,$$

  which implies, since $\lim_{x\to\infty}U(x)=0$, that

$$-\int_0^\infty x^{r+1}dU(x)=(r+1)\int_0^\infty y^r U(y)dy, \tag{30}$$

  which is positive and finite.

  Now, for $t>0$, we have, integrating by parts and using again the continuity of U,

$$t^{r+1}U(t)=(r+1)\int_0^t x^r U(x)dx+\int_0^t x^{r+1}dU(x)$$

  where the integrals on the right hand side of the equality are finite as $t\to\infty$ and their sum tends to 0 via (30). This implies that, $\forall r>0$, $t^{r+1}U(t)\to 0$ as $t\to\infty$.

  For $r\le 0$, we have, for $t\ge 1$, using the previous result, $t^{r+1}U(t)\le t^2U(t)\to 0$ as $t\to\infty$.

  This completes the proof that $U\in\mathcal{M}_\infty$.

- *Proof of (ii)-(a)*

  Suppose $U\in\mathcal{M}_{-\infty}$. Then, by definition (11), for any $\rho\in\mathbb{R}$, we have $\lim_{x\to\infty}\frac{U(x)}{x^\rho}=\infty$, which implies that for $c>0$, there exists $x_0>1$ such that, for all $x\ge x_0$, $U(x)\ge cx^\rho$, from which we deduce that, U being bounded on finite intervals,

$$\int_1^\infty x^{r-1}U(x)dx\ge c\int_{x_0}^\infty x^{r-1+\rho}dx$$

  which is infinite whenever $r\ge-\rho$.

  The argument applying for any ρ, we conclude that $\kappa_U=-\infty$.

- *Proof of (ii)-(b)*

  Let $r\ge 0$. We can write, for $s+2<0$ and $t>1$,

$$\begin{aligned}
0 &\ge -\int_1^t x^{s+1}d\left(x^rU(x)\right) \qquad (x^rU(x)\,\text{being non-decreasing})\\
&= \int_1^t\left(\int_x^t d\left(y^{s+1}\right)-t^{s+1}\right)d\left(x^rU(x)\right)\\
&= \int_1^t y^{s+1}\left(\int_1^y d\left(x^rU(x)\right)\right)dy-t^{s+1}\int_1^t d\left(x^rU(x)\right)\\
&= \int_1^t y^{s+r+1}U(y)dy-\frac{t^{s+2}-1}{s+2}U(1)-t^{s+1}\left(t^rU(t)-U(1)\right) \quad (\,U\text{ being continue}).
\end{aligned}$$

Hence we obtain, as $t \to \infty$, $t^{s+r+1}U(t) \to \infty$ since $\int_1^t y^{s+r+1}U(y)dy \to \infty$ and $\frac{t^{s+2}}{s+2} + t^{s+1} \to 0$ (under the assumption $s < -2$).
This implies that $U \in \mathcal{M}_{-\infty}$ since $s+r+1 \in \mathbb{R}$. □

Proof of Remark 1.6 -1.

Set $A = \int_1^\infty e^{-x}dx = e^{-1}$ and let us prove that $U \in \mathcal{M}_\infty$. If $r > 0$, then

$$\begin{aligned}\int_1^\infty x^r U(x)dx &\le A + \sum_{n=1}^\infty \int_n^{n+1/n^n} x^r U(x)dx = A + \sum_{n=1}^\infty \int_n^{n+1/n^n} x^{r-1}dx \\ &\le A + \sum_{n=1}^\infty \int_n^{n+1/n^n} x^{\lceil r\rceil -1}dx = A + \frac{1}{\lceil r\rceil}\sum_{n=1}^\infty \left((n+1/n^n)^{\lceil r\rceil} - n^{\lceil r\rceil}\right)dx \\ &= A + \frac{1}{\lceil r\rceil}\sum_{n=1}^\infty n^{-(n-1)\lceil r\rceil -1}\sum_{k=0}^{\lceil r\rceil -1}(1+1/n^{n-1})^k < \infty.\end{aligned}$$

If $r \le 0$, then we can write $\int_1^\infty x^r U(x)dx \le \int_1^\infty xU(x)dx$, which is finite using the previous result with $r = 1$.

Now, let us prove $U \notin \mathcal{M}_\infty$ by contradiction.

Suppose $U \in \mathcal{M}_\infty$. Then Theorem 1.4 implies that $\lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\infty$, which contradicts

$$\lim_{n\to\infty} \frac{\log(U(n))}{\log(n)} = \lim_{n\to\infty} \frac{\log(1/n)}{\log(n)} = -1 > -\infty.$$

□

Proof of Theorem 1.6.

- *Proof of (i)*

  Suppose $U \in \mathcal{M}_\infty$. By Theorem 1.4, we have $\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = \infty$. It implies that there exists $b > 1$ such that, for $x \ge b$, $\beta(x) := -\frac{\log(U(x))}{\log(x)} > 0$. Defining, for $x \ge b$, $\alpha(x) := \beta(x)\log(x)$, gives (i).

- *Proof of (ii)*

  Suppose $U \in \mathcal{M}_{-\infty}$. By Properties 1.3, (i), $1/U \in \mathcal{M}_\infty$. Applying the previous result to $1/U$ implies that there exists a positive function α satisfying $\alpha(x)/\log(x) \underset{x\to\infty}{\to} \infty$ such that $1/U(x) = \exp(-\alpha(x))$, $x \ge b$ for some $b > 1$. Hence we get $U(x) = \exp(-\alpha(x))$, $x \ge b$, as required.

- *Proof of (iii)*

  Assume that U satisfies, for $x \ge b$, $U(x) = \exp(-\alpha(x))$, for some $b > 1$ and α satisfying $\alpha(x)/\log(x) \underset{x\to\infty}{\to} \infty$. A straightforward computation gives $\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = \lim_{x\to\infty} \frac{\alpha(x)}{\log(x)} = \infty$. Hence $U \in \mathcal{M}_\infty$.

We can proceed exactly in the same way when supposing that U satisfies, for $x \geq b$, $U(x) = \exp(\alpha(x))$ for some $b > 1$ and α satisfying $\alpha(x)/\log(x) \underset{x\to\infty}{\to} \infty$, to conclude that $U \in \mathscr{M}_{-\infty}$. $\square$

Proof of Properties 1.3.

- *Proof of (i)*

  It is straightforward since, for $\rho \in \mathbb{R}$, $\lim\limits_{x\to\infty} \dfrac{U(x)}{x^\rho} = 0 \iff \lim\limits_{x\to\infty} \dfrac{1/U(x)}{x^{-\rho}} = \infty$.

- *Proof of (ii)*

  ▷ Suppose $(U, V) \in \mathscr{M}_{-\infty} \times \mathscr{M}$ with ρ_V defined in (2).
  Let $\epsilon > 0$. Writing $\dfrac{V(x)}{U(x)} = \dfrac{V(x)}{x^{\rho_V+\epsilon}} \left(\dfrac{U(x)}{x^{\rho_V+\epsilon}}\right)^{-1}$, we obtain $\lim\limits_{x\to\infty} \dfrac{V(x)}{U(x)} = 0$ since $V \in \mathscr{M}$ with ρ_V satisfying (2) and U satisfies (11) with $\rho_U = \rho_V + \epsilon \in \mathbb{R}$.

  ▷ Suppose $(U, V) \in \mathscr{M}_{-\infty} \times \mathscr{M}_\infty$.
  Let $\rho > 0$. We have $\lim\limits_{x\to\infty} \dfrac{V(x)}{U(x)} = \lim\limits_{x\to\infty} \dfrac{V(x)}{x^\rho} \left(\dfrac{U(x)}{x^\rho}\right)^{-1} = 0$ since V satisfies (10) and U (11).

  ▷ Suppose $(U, V) \in \mathscr{M} \times \mathscr{M}_\infty$ with ρ_U defined in (2).
  By Properties 1.1, (iv), and Properties 1.3, (i), we have $(1/U, 1/V) \in \mathscr{M} \times \mathscr{M}_{-\infty}$. The result follows because $\lim\limits_{x\to\infty} \dfrac{V(x)}{U(x)} = \lim\limits_{x\to\infty} \dfrac{1/U(x)}{1/V(x)} = 0$.

- The proof of (iii) is immediate. $\square$

Proof of Properties 1.4. Let U, $V \in \mathscr{M}$ with $\mathscr{M}$-index κ_U and κ_V respectively.

- *Proof of (i)*

  It is straightforward as $\lim\limits_{x\to\infty} \dfrac{\log(U(x)\,V(x))}{\log(x)} = \lim\limits_{x\to\infty} \left(\dfrac{\log(U(x))}{\log(x)} + \dfrac{\log(V(x))}{\log(x)}\right)$.

- *Proof of (ii)*

  We distinguish the next three cases.
  (a) *Let $U \in \mathscr{M}_\infty$ and $V \in \mathscr{M}$ with $\rho_V \notin [-1, 0)$.*

  Let $W(x) = x^\eta 1_{(x\geq 1)} + 1_{(0<x<1)}$, with $\eta = -2$ if $\rho_V \geq 0$, or $\eta = \rho_V - 1$ if $\rho_V < -1$. Note that $W \in \mathscr{M}$ with $\rho_W = \eta < \rho_V$.

  By Properties 1.3, (ii), $\lim\limits_{x\to\infty} \dfrac{U(x)}{W(x)} = 0$, so for $0 < \delta < 1$, there exists $x_0 \geq 1$ such that, for all $x \geq x_0$, $U(x) \leq \delta W(x)$.

  Consider Z defined by $Z(x) = U(x)1_{(0<x<x_0)} + W(x)1_{(x\geq x_0)}$, which satisfies $Z \geq U$ and $Z \in \mathscr{M}$ with $\rho_Z = \rho_W = \eta < \rho_V$. Applying Properties 1.2, (ii), gives $Z * V \in \mathscr{M}$ with $\rho_{Z*V} = \rho_Z \vee \rho_V = \rho_V$ (note that the restriction on ρ_v corresponds to the condition given in Properties 1.2, (ii)).

We deduce that, for any $x>0$, $U*V(x)\le Z*V(x)$, and, for $\epsilon>0$,

$$\frac{U*V(x)}{x^{\rho_V+\epsilon}}\le\frac{Z*V(x)}{x^{\rho_V+\epsilon}}\underset{x\to\infty}{\to}0.$$

Moreover, applying Fatou's Lemma gives

$$\lim_{x\to\infty}\frac{U*V(x)}{x^{\rho_V-\epsilon}}\ge\lim_{x\to\infty}\int_0^1 U(t)\frac{V(x-t)}{x^{\rho_V-\epsilon}}dt\ge\underline{\lim}_{x\to\infty}\int_0^1 U(t)\frac{V(x-t)}{x^{\rho_V-\epsilon}}dt\ge\int_0^1 U(t)\lim_{x\to\infty}\left(\frac{V(x-t)}{x^{\rho_V-\epsilon}}\right)dt=\infty.$$

Therefore, $U*V\in\mathcal{M}$ with $\mathcal{M}$-index $\rho_{U*V}=\rho_V$.

(b) $(U,V)\in\mathcal{M}_\infty\times\mathcal{M}_\infty$, *then* $U*V\in\mathcal{M}_\infty$

Let $\rho\in\mathbb{R}$.

Consider $U\in\mathcal{M}_\infty$. We have, applying Theorem 1.4, $\lim\limits_{x\to\infty}\dfrac{\log(U(x))}{\log(x)}=-\infty$. Rewriting this limit as

$$\lim_{x\to\infty}\frac{\log(U(x))}{\log(1/x)}=\infty$$

we deduce that, for $c\ge|\rho|+1>0$, there exists $x_U>1$ such that, for $x\ge x_U$, $\log(U(x))\le c\log(1/x)$, i.e. $U(x)\le x^{-c}$. On $V\in\mathcal{M}_\infty$, a similar reasoning leads to that there exists $x_V>1$ such that, for $x\ge x_V$, $V(x)\le x^{-c}$.

Using the change of variable $s=x-t$, we have, $\forall\ x\ge 2\max(x_U,x_V)>0$,

$$\begin{aligned}\frac{U*V(x)}{x^\rho}&=\int_0^{x/2}U(t)\frac{V(x-t)}{x^\rho}dt+\int_{x/2}^{x}U(t)\frac{V(x-t)}{x^\rho}dt\\&\le\frac{1}{x^{\rho+c}}\int_0^{x/2}U(t)\Big(1-\frac{t}{x}\Big)^{-c}dt+\frac{1}{x^{\rho+c}}\int_0^{x/2}V(s)\Big(1-\frac{s}{x}\Big)^{-c}ds\\&\le\frac{2^c}{x^{\rho+c}}\int_0^{x/2}U(t)dt+\frac{2^c}{x^{\rho+c}}\int_0^{x/2}V(s)ds,\end{aligned}$$

since, for $0\le t\le x/2$, *i.e.* $0<\dfrac{1}{2}\le 1-\dfrac{t}{x}\le 1$, $\Big(1-\dfrac{t}{x}\Big)^{-c}\le 2^c$.

This implies, via the integrability of U and V, for $\rho\in\mathbb{R}$, $\lim\limits_{x\to\infty}\dfrac{U*V(x)}{x^\rho}=0$. Hence $U*V\in\mathcal{M}_\infty$.

(c) *Let* $U\in\mathcal{M}_{-\infty}$ *and* $V\in\mathcal{M}$ *or* $\mathcal{M}_{\pm\infty}$.

We apply Fatou's Lemma, as in (a), to obtain, for any $\rho\in\mathbb{R}$,

$$\lim_{x\to\infty}\frac{U*V(x)}{x^\rho}\ge\underline{\lim}_{x\to\infty}\int_0^1 V(t)\frac{U(x-t)}{x^\rho}dt\ge\int_0^1 V(t)\lim_{x\to\infty}\left(\frac{U(x-t)}{x^\rho}\right)dt=\infty.$$

We conclude that $U*V\in\mathcal{M}_{-\infty}$.

- *Proof of (iii)*

  First, note that if $V\in\mathcal{M}_{-\infty}$, then $\lim\limits_{x\to\infty}V(x)=\infty$. Hence writing

$$\frac{\log(U(V(x)))}{\log(x)}=\frac{\log\big(U(y)\big)}{\log(y)}\times\frac{\log(V(x))}{\log(x)},\quad\text{with } y=V(x)$$

  allows one to conclude.

$\square$

A.3 Proofs of results concerning $\mathcal{O}$

Proof of Example 1.5.

Let $x \in [x_n, x_{n+1})$, $n \geq 1$. We can write

$$\frac{\log(U(x))}{\log(x)} = \frac{\log\left(x_n^{\alpha(1+\beta)}\right)}{\log(x)} = \alpha(1+\beta)\frac{\log(x_n)}{\log(x)}. \tag{31}$$

Since $\log(x_n) \leq \log(x) < \log(x_{n+1}) = (1+\alpha)\log(x_n)$, we obtain

$$\frac{\alpha(1+\beta)}{1+\alpha} < \frac{\log(U(x))}{\log(x)} \leq \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0,$$

and

$$\alpha(1+\beta) \leq \frac{\log(U(x))}{\log(x)} < \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta < 0,$$

from which we deduce

$$\mu(U) \geq \frac{\alpha(1+\beta)}{1+\alpha} \quad \text{and} \quad \nu(U) \leq \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0,$$

and

$$\mu(U) \geq \alpha(1+\beta) \quad \text{and} \quad \nu(U) \leq \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta < 0.$$

Moreover, taking $x = x_n$ in (31) leads to

$$\lim_{n\to\infty} \frac{\log(U(x_n))}{\log(x_n)} = \alpha(1+\beta)$$

which implies

$$\nu(U) \geq \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0$$

and

$$\mu(U) \leq \alpha(1+\beta), \quad \text{if} \quad 1+\beta < 0.$$

Hence, to conclude, it remains to prove that

$$\mu(U) \leq \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta > 0, \quad \text{and} \quad \nu(U) \geq \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta < 0.$$

If $1+\beta > 0$, the function $\log(U(x))/\log(x)$ is strictly decreasing continuous on $(x_n; x_{n+1})$ reaching the supremum value $\alpha(1+\beta)$ and the infimum value $\alpha(1+\beta)/(1+\alpha)$. Hence, for $\delta > 0$ such that

$$\frac{\alpha(1+\beta)}{1+\alpha} < \frac{\alpha(1+\beta)}{1+\alpha} + \delta < \alpha(1+\beta),$$

there exists $x_n < y_n < x_{n+1}$ satisfying

$$\frac{\log\big(U(y_n)\big)}{\log(y_n)} = \frac{\alpha(1+\beta)}{1+\alpha} + \delta.$$

Since $y_n \to \infty$ as $n \to \infty$ because $x_n \to \infty$ as $n \to \infty$, $\mu(U) \le \lim_{n\to\infty} \frac{\log\big(U(y_n)\big)}{\log(y_n)} = \frac{\alpha(1+\beta)}{1+\alpha} + \delta$ follows. Hence we conclude $\mu(U) \le \frac{\alpha(1+\beta)}{1+\alpha}$ since δ is arbitrary.

If $1+\beta < 0$, a similar development to the case $1+\beta > 0$ allows proving $\nu(U) \ge \frac{\alpha(1+\beta)}{1+\alpha}$.

Moreover, if $1+\beta<0$ we have that U is a tail of distribution. Let us check that the rv having a tail of distribution $\overline{F} = U$ has a finite sth moment whenver $0 \le s < -\alpha(1+\beta)/(1+\alpha)$.

Let $s \ge 0$. We have

$$\int_0^\infty x^s dF(x) = \sum_{n=1}^\infty x_n^s \left(U(x_n^-) - U(x_n^+)\right)$$
$$= \sum_{n=2}^\infty x_n^s \left(x_{n-1}^{\alpha(1+\beta)} - x_n^{\alpha(1+\beta)}\right) = \sum_{n=2}^\infty x_n^s \left(x_n^{\frac{\alpha(1+\beta)}{1+\alpha}} - x_n^{\alpha(1+\beta)}\right) \le \sum_{n=2}^\infty x_n^{s+\frac{\alpha(1+\beta)}{1+\alpha}} < \infty$$

because $s < -\alpha(1+\beta)/(1+\alpha)$.

Note that if $s \ge -\alpha(1+\beta)/(1+\alpha)$, $\int_0^\infty x^s dF(x) = \infty$. $\square$

Proof of Example 1.6.

If $\alpha > 0$, $\nu(U) = \infty$ comes from

$$\nu(U) = \overline{\lim_{x\to\infty}} \frac{\log(U(x))}{\log(x)} \ge \lim_{x_n\to\infty} \frac{\log(U(x_n))}{\log(x_n)} = \lim_{x_n\to\infty} \frac{\alpha x_n \log(2)}{\log(x_n)} = \infty,$$

and, if $\alpha < 0$, $\mu(U) = -\infty$ comes from

$$\mu(U) = \underline{\lim_{x\to\infty}} \frac{\log(U(x))}{\log(x)} \le \lim_{x_n\to\infty} \frac{\log(U(x_n))}{\log(x_n)} = \lim_{x_n\to\infty} \frac{\alpha x_n \log(2)}{\log(x_n)} = -\infty.$$

Next, let $\epsilon > 0$ be small enough. Then, we have, if $\alpha > 0$,

$$\begin{aligned}\mu(U) &= \underline{\lim_{x\to\infty}} \frac{\log(U(x))}{\log(x)} \le \lim_{x_n\to\infty} \frac{\log(U(x_n-\epsilon))}{\log(x_n-\epsilon)} \\ &= \lim_{x_n\to\infty} \frac{\log(2^{\alpha x_{n-1}})}{\log(2^{x_{n-1}/c})} \frac{\log(2^{x_{n-1}/c})}{\log(2^{x_{n-1}/c}-\epsilon)} = \lim_{x_n\to\infty} \frac{\log(2^{\alpha x_{n-1}})}{\log(2^{x_{n-1}/c})} = \alpha c,\end{aligned}$$

and, if $\alpha < 0$,

$$\begin{aligned}\nu(U) &= \overline{\lim_{x\to\infty}} \frac{\log(U(x))}{\log(x)} \ge \lim_{x_n\to\infty} \frac{\log(U(x_n-\epsilon))}{\log(x_n-\epsilon)} \\ &= \lim_{x_n\to\infty} \frac{\log(2^{\alpha x_{n-1}})}{\log(2^{x_{n-1}/c})} \frac{\log(2^{x_{n-1}/c})}{\log(2^{x_{n-1}/c}-\epsilon)} = \lim_{x_n\to\infty} \frac{\log(2^{\alpha x_{n-1}})}{\log(2^{x_{n-1}/c})} = \alpha c.\end{aligned}$$

It remains to prove that, if $\alpha > 0$, $\mu(U) \ge \alpha c$, and, if $\alpha < 0$, $\nu(U) \le \alpha c$. It follows from the fact that, for $x_n \le x < x_{n+1}$,

$$\frac{\log(U(x))}{\log(x)} = \alpha \frac{x_n \log(2)}{\log(x)} = \alpha c \frac{\log(x_{n+1})}{\log(x)} \begin{cases} > & \alpha c, \quad \text{if } \alpha > 0 \\ < & \alpha c, \quad \text{if } \alpha < 0. \end{cases}$$

Next, if $\alpha < 0$ we have that U is a tail of distribution. Let us check that the rv having a tail of distribution $\overline{F} = U$ has a finite sth moment whenever $0 \le s < -\alpha c$.

Let $s > 0$ and denote $x_0 = 0$. We have

$$\int_0^\infty x^s dF(x) = \sum_{n=1}^\infty x_n^s \left(U(x_n^-) - U(x_n^+)\right) = \sum_{n=1}^\infty x_n^s \left(2^{\alpha x_{n-1}} - 2^{\alpha x_n}\right) \le \sum_{n=1}^\infty 2^{(s/c-\alpha)x_{n-1}} < \infty,$$

because $s < -\alpha c$.

If $s = 0$, let $\epsilon = -\alpha c/2$ (> 0) and the statement follows from $\int_0^\infty dF(x) = \int_0^1 dF(x) + \int_1^\infty dF(x) \le \int_0^1 dF(x) + \int_1^\infty x^\alpha dF(x) < \infty$.

Note that if $s \ge -\alpha c$, $\int_0^\infty x^s dF(x) = \infty$. □

B Proofs of results given in Section 2

B.1 Section 2.1

Let us introduce the following functions that will be used in the proofs.

We define, for some $b > 0$ and $r \in \mathbb{R}$,

$$V_r(x) = \begin{cases} \int_b^x y^r U(y) dy, & x \ge b \\ 1, & 0 < x < b \end{cases} \quad ; \quad W_r(x) = \begin{cases} \int_x^\infty y^r U(y) dy, & x \ge b \\ 1, & 0 < x < b. \end{cases} \tag{32}$$

For the main result, we will need the following lemma which is of interest on its own.

Lemma B.1. *Let $U \in \mathcal{M}$ with finite $\mathcal{M}$-index κ_U and let $b > 0$.*

- *(i) Consider V_r defined in (32) with $r + 1 > \kappa_U$. Then $V_r \in \mathcal{M}$ and its $\mathcal{M}$-index κ_{V_r} satisfies $\kappa_{V_r} = \kappa_U - (r+1)$.*
- *(ii) Consider W_r defined in (32) with $r + 1 < \kappa_U$. Then $W_r \in \mathcal{M}$ and its $\mathcal{M}$-index κ_{W_r} satisfies $\kappa_{W_r} = \kappa_U - (r+1)$.*

Proof of Theorem 2.2.

- *Proof of the necessary condition of (K1*)*

  As an immediate consequence of Lemma B.1, (i), we have, assuming that $\rho + r > 0$:

  $$\begin{aligned} &U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_U = -\rho \text{ such that } (r-1)+1 = r > -\rho = \kappa_U \\ &\implies \quad V_{r-1}(x) = \int_b^x t^{r-1} U(t) dt \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_{V_{r-1}} = \kappa_U - r = -\rho - r \end{aligned}$$

  Hence, by applying Theorems 1.1 and 1.2 to V_{r-1}, the result follows:

  $$\lim_{x\to\infty} \frac{\log\left(\int_b^x t^{r-1} U(t) dt\right)}{\log(x)} = \lim_{x\to\infty} \frac{\log(V_{r-1}(x))}{\log(x)} = -\kappa_{V_{r-1}} = \rho + r > 0.$$

- *Proof of the sufficient of (K1*)*

  Using $(C1r)$ and $\lim_{x\to\infty} \frac{\log\left(\int_b^x t^{r-1}U(t)dt\right)}{\log(x)} = \rho + r$ gives

$$\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = \lim_{x\to\infty} -\frac{\log\left(\frac{x^r U(x)}{\int_b^x t^{r-1}U(t)dt}\right) + \log\left(x^{-r}\int_b^x t^{r-1}U(t)dt\right)}{\log(x)}$$
$$= r + \lim_{x\to\infty} -\frac{\log\left(\int_b^x t^{r-1}U(t)dt\right)}{\log(x)} = r - (\rho + r) = -\rho,$$

  and the statement follows.

- *Proof of the necessary condition of (K2*)*

  As an immediate consequence of Lemma B.1, (ii), we have, assuming that $\rho + r < 0$:

$$U \in \mathscr{M} \text{ with } \mathscr{M}\text{-index } \kappa_U = -\rho \text{ such that } (r-1)+1 = r < -\rho = \kappa_U$$
$$\implies \quad W_{r-1}(x) = \int_x^\infty t^{r-1}U(t)dt \in \mathscr{M} \text{ with } \mathscr{M}\text{-index } \kappa_{W_{r-1}} = \kappa_U - r = -\rho - r$$

  Hence, by applying Theorems 1.1 and 1.2 to W_{r-1}, the result follows:

$$\lim_{x\to\infty} \frac{\log\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\log(x)} = \lim_{x\to\infty} \frac{\log(W_{r-1}(x))}{\log(x)} = -\kappa_{W_{r-1}} = \rho + r < 0.$$

- *Proof of the sufficient of (K2*)*

  Using $(C2r)$ and $\lim_{x\to\infty} \frac{\log\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\log(x)} = \rho + r$ gives

$$\lim_{x\to\infty} -\frac{\log(U(x))}{\log(x)} = \lim_{x\to\infty} -\frac{\log\left(\frac{x^r U(x)}{\int_x^\infty t^{r-1}U(t)dt}\right) + \log\left(x^{-r}\int_x^\infty t^{r-1}U(t)dt\right)}{\log(x)}$$
$$= r + \lim_{x\to\infty} -\frac{\log\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\log(x)} = r - (\rho + r) = -\rho$$

  and the statement follows.

- *Proof of the necessary condition of (K3*); case* $\int_b^\infty t^{r-1}U(t)dt = \infty$ *with* $b > 1$.

  On one hand, assumed $\rho + r = 0$, $U \in \mathscr{M}$ with $\mathscr{M}$-index $\kappa_U = -\rho$ implies, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty \tag{33}$$

  On the other hand, $\int_b^\infty t^{r-1}U(t)dt = \infty$ implies $\lim_{x\to\infty}\int_b^x t^{r-1}U(t)dt = \infty$. Hence we can apply the L'Hôpital's rule to the first limit of (33) to get, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{\int_b^x t^{r-1}U(t)dt}{x^\epsilon} = \lim_{x\to\infty} \frac{x^{r-1}U(x)}{\epsilon x^{-1+\epsilon}} = \lim_{x\to\infty} \frac{U(x)}{\epsilon x^{-r-1+\epsilon}} = \lim_{x\to\infty} \frac{U(x)}{\epsilon x^{\rho+\epsilon}} = 0 \tag{34}$$

Moreover, we have, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{\int_b^x t^{r-1} U(t)dt}{x^{-\epsilon}} = \left(\lim_{x\to\infty} \int_b^x t^{r-1} U(t)dt\right) \left(\lim_{x\to\infty} x^{\epsilon}\right) = \infty \times \infty = \infty \tag{35}$$

Defining V_{r-1} as in (32) we deduce from (34) and (35) that $V_{r-1} \in \mathscr{M}$ with $\mathscr{M}$-index $0 = \rho + r$. So, taking $x \geq b$, the required result follows:

$$\lim_{x\to\infty} \frac{\log\left(\int_b^x t^{r-1} U(t)dt\right)}{\log(x)} = \lim_{x\to\infty} \frac{\log(V_{r-1}(x))}{\log(x)} = \rho + r = 0$$

- *Proof of the necessary condition of (K3*); case* $\int_b^\infty t^{r-1} U(t)dt < \infty$ *with* $b > 1$.

  Suppose $U \in \mathscr{M}$ with $\mathscr{M}$-index $\kappa_U = -\rho$. By a straightforward computation we have

$$\lim_{x\to\infty} \frac{\log\left(\int_b^x t^{r-1} U(t)dt\right)}{\log(x)} = \frac{\log\left(\int_b^\infty t^{r-1} U(t)dt\right)}{\lim_{x\to\infty} \log(x)} = 0 = \rho + r$$

- *Proof of the sufficient condition of (K3*)*

  A similar proof used to prove the sufficient condition of (K1*).

□

Proof of Lemma B.1.

- *Proof of (i)*

  Let us prove that V_r defined in (32) belongs to $\mathscr{M}$ with $\mathscr{M}$-index $\kappa_{V_r} = \kappa_U - (r+1)$.

  Choose $\rho = -\kappa_U + r + 1 > 0$ and $0 < \epsilon < \rho$. Note that $x^{\rho\pm\epsilon} \to \infty$ as $\rho \pm \epsilon > 0$.

  Combining, for $x > 1$, under the assumption $r + 1 > \kappa_U$, and for $U \in \mathscr{M}$,

$$\lim_{x\to\infty} V_r(x) = \int_b^1 y^r U(y)dy + \int_1^\infty y^r U(y)dy = \infty,$$

  and,

$$\lim_{x\to\infty} \frac{(V_r(x))'}{\left(x^{\rho+\delta}\right)'} = \lim_{x\to\infty} \frac{U(x)}{(\rho+\delta)x^{-\kappa_U+\delta}} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

  provides, applying the L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{V_r(x)}{x^{\rho+\delta}} = \lim_{x\to\infty} \frac{(V_r(x))'}{\left(x^{\rho+\delta}\right)'} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

  which implies that $V_r \in \mathscr{M}$ with $\mathscr{M}$-index $\kappa_{V_r} = -\rho = \kappa_U - (r+1)$, as required.

- *Proof of (ii)*

  First let us check that W_r is well-defined. Let $\delta = (\kappa_U - r - 1)/2$ (> 0 by assumption). We have, for $U \in \mathscr{M}$, $\lim_{x\to\infty} \frac{U(x)}{x^{-\kappa_U+\delta}} = 0$, which implies that for $c > 0$ there exists $x_0 \geq 1$ such that for all $x \geq x_0$, $\frac{U(x)}{x^{-\kappa_U+\delta}} \leq c$.

Hence, one has, $\forall\ x \ge x_0$,

$$\int_x^{\infty} y^r U(y) dy \le c \int_x^{\infty} y^{-\kappa_U+\delta+r} dy = c \int_x^{\infty} y^{\frac{-\kappa_U+r+1}{2}-1} dy < \infty$$

because of $-\kappa_U + r + 1 < 0$. Then, we can conclude, U being bounded on finite intervals, that W_r is well-defined.

Now choose $\rho = -\kappa_U + r + 1 < 0$ and $0 < \epsilon < -\rho$. We have $x^{\rho\pm\epsilon} \to 0$ as $\rho \pm \epsilon < 0$. We will proceed as in (i).

For $x > 1$, under the assumption $r + 1 < \kappa_U$, for $U \in \mathcal{M}$, we have $\lim_{x\to\infty} W_r(x) = \int_x^{\infty} y^r U(y) dy = 0$, and $\lim_{x\to\infty} \frac{(W_r(x))'}{\left(x^{\rho+\delta}\right)'} = \lim_{x\to\infty} -\frac{U(x)}{(\rho+\delta)x^{-\kappa+\delta}} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon \end{cases}$.

Hence applying the L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{W_r(x)}{x^{\rho+\delta}} = \lim_{x\to\infty} \frac{(W_r(x))'}{\left(x^{\rho+\delta}\right)'} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

which implies that $W_r \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_{W_r} = -\rho = \kappa_U - (r+1)$. □

B.2 Section 2.2

Proof of Theorem 2.4.

- *Proof of (i)*

  Changing the order of integration in (23), using the continuity of U and the assumption $U(0^+) = 0$, give, for $s > 0$,

  $$\widehat{U}(s) = s \int_{(0;\infty)} e^{-xs} U(x) dx,$$

  or, with the change of variable $y = x/s$,

  $$\widehat{U}\left(\frac{1}{s}\right) = \int_{(0;\infty)} e^{-y} U(sy) dy.$$

  Let $U \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha) < 0$. Let $0 < \epsilon < \alpha$.

  We have, via Theorems 1.1 and 1.2, that there exists $x_0 > 1$ such that, for $x \ge x_0$,

  $$x^{\alpha-\epsilon} \le U(x) \le x^{\alpha+\epsilon}.$$

  Hence, for $s > 1$, we can write

  $$\int_{x_0/s}^{\infty} e^{-x}(xs)^{\alpha-\epsilon} dx \le \int_{x_0/s}^{\infty} e^{-x} U(xs) dx \le \int_{x_0/s}^{\infty} e^{-x}(xs)^{\alpha+\epsilon} dx$$

  so $\frac{\int_0^{x_0/s} e^{-x} U(xs) dx + \int_{x_0/s}^{\infty} e^{-x} x^{\alpha-\epsilon} dx}{s^{-\alpha+\epsilon}} \le \widehat{U}\left(\frac{1}{s}\right) \le \frac{\int_0^{x_0/s} e^{-x} U(xs) dx + \int_{x_0/s}^{\infty} e^{-x} x^{\alpha+\epsilon} dx}{s^{-\alpha-\epsilon}}$,

  from which we deduce that

  $$-\alpha - \epsilon \le \lim_{s\to\infty} -\frac{\log\left(\widehat{U}(1/s)\right)}{\log(s)} \le -\alpha + \epsilon.$$

Then we obtain, ϵ being arbitrary, $\lim_{s\to\infty} -\dfrac{\log\left(\widehat{U}(1/s)\right)}{\log(s)} = -\alpha$.

The conclusion follows, applying Theorem 1.1, to get $\widehat{U}\circ g \in \mathcal{M}$ with $g(s) = 1/s$, $(s>0)$, and, Theorem 1.2, for the $\mathcal{M}$-index.

- *Proof of (ii)*

  Let $0 < \epsilon < \alpha$.

  Since we assumed $U(0^+) = 0$, we have, for $s > 1$,

$$e^{-1}U(s) \le \int_{(0;s)} e^{-\frac{x}{s}}\, dU(x) \le \int_{(0;\infty)} e^{-\frac{x}{s}}\, dU(x) = \widehat{U}\left(\frac{1}{s}\right). \tag{36}$$

  Changing the order of integration in the last integral (on the right hand side of the previous equation), and using the continuity of U and the fact that $U(0^+) = 0$, gives, for $s > 0$,

$$\widehat{U}\left(\frac{1}{s}\right) = \int_{(0;\infty)} e^{-x}U(sx)\,dx. \tag{37}$$

  Set $I_\eta = \displaystyle\int_{(0;\infty)} e^{-x}x^\eta\,dx$, for $\eta \in [0,\alpha)$ (such that $x^{-\eta}U(x)$ concave, by assumption). Introducing the function $V(x) := I_\eta\,(sx)^{-\eta}\,U(sx)$, which is concave, and the rv Z having the probability density function defined on $\mathbb{R}^+$ by $e^{-x}x^\eta/I_\eta$, we can write

$$\int_{(0;\infty)} e^{-x}U(sx)\,dx = s^\eta \int_{(0;\infty)} V(x)\,\frac{e^{-x}x^\eta}{I_\eta}\,dx = s^\eta\, E[V(Z)] \le s^\eta\, V\left(E[Z]\right),$$

  applying Jensen's inequality. Hence we obtain, using that $E[Z] = I_{\eta+1}/I_\eta$ and the definition of V,

$$\int_{(0;\infty)} e^{-x}U(sx)\,dx \le \frac{I_\eta^{\eta+1}}{I_{\eta+1}^{\eta}}\, U\left(s\, I_{\eta+1}/I_\eta\right),$$

  from which we deduce, using (37), that

$$\frac{1}{s^{\alpha-\epsilon}}\widehat{U}\left(\frac{1}{s}\right) \le \frac{I_\eta^{\eta+1-\alpha+\epsilon}}{I_{\eta+1}^{\eta-\alpha+\epsilon}} \times \frac{U\left(s\, I_{\eta+1}/I_\eta\right)}{\left(s\, I_{\eta+1}/I_\eta\right)^{\alpha-\epsilon}}.$$

  Therefore, since $\widehat{U}\circ g \in \mathcal{M}$ with $g(s) = 1/s$ and $\mathcal{M}$-index $(-\alpha)$, we obtain

$$\frac{I_\eta^{\eta+1-\alpha+\epsilon}}{I_{\eta+1}^{\eta-\alpha+\epsilon}} \times \frac{U\left(s\, I_{\eta+1}/I_\eta\right)}{\left(s\, I_{\eta+1}/I_\eta\right)^{\alpha-\epsilon}} \underset{s\to\infty}{\longrightarrow} \infty.$$

  But $\widehat{U}\circ g \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$ also implies in (36) that $\dfrac{e^{-1}U(s)}{s^{\alpha+\epsilon}} \underset{s\to\infty}{\longrightarrow} 0$.
  From these last two limits, we obtain that $U \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$. □